Lecture 11

plan: §4.6 Variation of parameters

Recall in Lecture 9, we learned the undetermined coefficients method to find a particular soln to

$$ay'' + by' + cy = f(x)$$

Today we will introduce a more general and more powerful (But sometimes might requires more computations) method —— Variation of parameters.

Consider

$$ay'' + by' + cy = f(x) \qquad (1)$$

Suppose y_1, y_2 are two linear independent solns to the associated homogeneous eqn:

$$\begin{cases} ay_1'' + by_1' + cy_1 = 0 \\ ay_2'' + by_2' + cy_2 = 0 \end{cases} \Leftarrow \quad ay'' + by' + cy = 0 \qquad (2)$$

$\Rightarrow$ We know "$c_1 y_1 + c_2 y_2$" gives the general solns to D.E (2).

The idea of variation of parameters:

replace the constants c_1, c_2 by two functions $v_1(x), v_2(x)$, then "$c_1 y_1 + c_2 y_2$" becomes

"$v_1 y_1 + v_2 y_2$"

and we seek a particular soln of the form as above,

$$y_p = v_1(x)\, y_1(x) + v_2(x)\, y_2(x) \qquad (*)$$

Suppose $(*)$ is a soln to (1). Then we plug $(*)$ into (1) $\Rightarrow$

$$a y_p'' + b y_p' + c y_p = f(x) \qquad (3)$$

Note
$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$
$$= (v_1' y_1 + v_2' y_2) + (v_1 y_1' + v_2 y_2')$$

Another key idea:

We impose the requirement

$$v_1' y_1 + v_2' y_2 = 0 \qquad (4)$$

(This will simplify our computation significantly)

Under the assumption (4), we have

$$y_p' = v_1 y_1' + v_2 y_2'$$

$$\Rightarrow \quad y_p'' = v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''$$

Substituting y_p, y_p', y_p'' into (3) $\Rightarrow$

LHS of (3)

$$= a(v_1' y_1' + v_2' y_2' + v_2 y_1'' + v_2 y_2'') + b(v_1 y_1' + v_2 y_2')$$
$$+ c(v_1 y_1 + v_2 y_2)$$

$$= \underbrace{a(v_1' y_1' + v_2' y_2')}_{(\mathrm{I})} + \underbrace{v_1 \overbrace{(a y_1'' + b y_1' + c y_1)}^{=0}}_{(\mathrm{II})} + \underbrace{v_2 \overbrace{(a y_2'' + b y_2' + c y_2)}^{=0}}_{(\mathrm{III})}$$

$$= a(v_1' y_1' + v_2' y_2') \leftarrow$$ need to make this equal to RHS of (3) $= f(x)$

To summarize, we need find v_1, v_2 that satisfy

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ a(v_1' y_1' + v_2' y_2') = f \end{cases} \qquad (4)$$

$$\Longleftrightarrow \begin{cases} v_1' y_1 + v_2' y_2 = 0 & ① \\ v_1' y_1' + v_2' y_2' = \frac{f}{a} & ② \end{cases}$$

Unknowns: v_1', v_2'

$y_2' \times ① - y_2 \times ② \Rightarrow$

$$\underbrace{(y_1 y_2' - y_2 y_1')}_{W(y_1, y_2)} v_1' = \frac{-f}{a} y_2$$

Recall

- The Wronskian of y_1, y_2:

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$.

- y_1, y_2 are Linearly indep. $\Longleftrightarrow W(y_1, y_2) \neq 0$

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$$\Rightarrow \quad v_1' = \frac{-f y_2}{a W(y_1, y_2)}$$

By ①,

$$v_2' = \frac{-v_1' y_1}{y_2} = \frac{f y_1}{a W(y_1, y_2)}$$

Finally we have the formulas for v_1, v_2:

$$v_1 = \int v_1' dx = \int \frac{-f y_2}{a W(y_1, y_2)} dx \quad ; \quad v_2 = \int v_2' dx = \int \frac{f y_1}{a W(y_1, y_2)} dx$$

Note in the above argument, a is NOT necessarily a constant, it can be a function of x, $a(x)$.

We summarize the above to the following:

Algorithm of variation of parameters method:

Find a particular soln to

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = f(x) \quad (5)$$

Step 1: Find two linearly independent solns $y_1(x)$, $y_2(x)$ to the homogeneous eqn

$$a(x)\, y'' + b(x)\, y' + c(x) y = 0$$

Step 2: Compute

$$V_1(x) = \int \frac{-f y_2}{a W(y_1, y_2)}\, dx$$

$$V_2(x) = \int \frac{f y_1}{a W(y_1, y_2)}\, dx$$

Step 3: Obtain a particular soln

$$y_p = v_1 y_1 + v_2 y_2$$

E.g ① Find a particular soln to

$$y'' + 2y' + y = \underbrace{e^{-x}}_{f(x)}$$

$a = 1$
$b = 2$
$c = 1$

② Find the general solns to

$$y'' + 2y' + y = e^{-x}$$

A: ① use ~~Variation of parameters~~

Step 1: Find two Linearly indep. solns y_1, y_2 to

$$y'' + 2y' + y = 0$$

Note the Char. eqn. is $\lambda^2 + 2\lambda + 1 = 0$

$$\Leftrightarrow \quad (\lambda+1)^2 = 0$$

$$\Leftrightarrow \quad \lambda_1 = \lambda_2 = -1$$

$\Rightarrow$ Two linearly indep. solns:

"$y_1 = e^{-x}$", "$y_2 = x e^{-x}$"

Step 2: Compute V_1, V_2

$$V_1 = \int \frac{-f y_2}{a W(y_1, y_2)} dx$$

Note $f(x) = e^{-x}$

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$a = 1$

First $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$$= (e^{-x})(xe^{-x})' - (xe^{-x})(e^{-x})'$$

$$= \underset{\nearrow \text{E.x.}}{\cdots} = e^{-2x}$$

$$\Rightarrow \quad V_1 = \int \frac{-\overset{f}{\underline{e^{-x}}}\ \overset{y_2}{\underline{(xe^{-x})}}}{\underset{a}{\underline{1}} \cdot \underset{W}{\underline{e^{-2x}}}} dx$$

$$= \int -x\,dx = -\frac{1}{2}x^2 + c$$

Since we only need one particular soln,

choose $c = 0$, $\Rightarrow V_1 = -\frac{1}{2}x^2$

Similarly

$$V_2 = \int \frac{f y_1}{a W(y_1, y_2)} dx$$

$$= \int \frac{\underline{e^{-x}}\ \underline{e^{-x}}}{\underline{1} \cdot \underline{e^{-2x}}} dx = \int 1 \cdot dx$$

$$= x + c$$

choose $c = 0 \Rightarrow v_2 = x$

Hence $\begin{cases} v_1 = -\frac{1}{2}x^2 \\ v_2 = x \end{cases}$

Step 3: obtain a particular soln:

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\frac{1}{2}x^2 e^{-x} + x \cdot x e^{-x}$$

$$= \frac{1}{2}x^2 e^{-x}$$

② Find the general solns:

$$y = y_p + y_h$$

$$= \frac{1}{2}x^2 e^{-x} + C_1 e^{-x} + C_2 x e^{-x}$$

Here y_h is the general solns to the homogeneous eqn $C_1 y_1 + C_2 y_2$

E.g.: Let $y_1 = x^2$, $y_2 = x^3$.

① Verify y_1, y_2 are both solns to

$$x^2 y'' - 4xy' + 6y = 0 \qquad (6)$$

② Compute and check $W(y_1, y_2) \neq 0$.

③ Find a particular soln to

$$\underbrace{x^2}_{a(x)} y'' \underbrace{-4x}_{b(x)} y' + \underbrace{6}_{c(x)} y = \underbrace{4x^3}_{f(x)} \qquad (7)$$

A: ① Check $y_1 = x^2$:

$$y_1' = 2x, \quad y_1'' = 2$$

plug y_1 in (6) $\Rightarrow$ LHS of (6) $= x^2 y_1'' - 4x y_1' + 6 y_1$

$$= 2x^2 - 4x \cdot 2x + 6x^2$$

$$= 0 = \text{RHS of } (6)$$

$\Rightarrow$ y_1 is a soln to (6).

check $y_2 = x^3$: Ex.

$y_1 = x^2 \quad y_2 = x^3$

② Compute :

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$
$$= x^2 (x^3)' - x^3 (x^2)'$$
$$= 3x^2 \cdot x^2 - x^3(2x) = x^4 \neq 0$$

$\Rightarrow$ y_1, y_2 are L.I.

③ Find a particular soln to

$$x^2 y'' - 4xy' + 6y = 4x^3 \qquad (7)$$

Step 1: Find two L.I solns to the homog. eqn

$$x^2 y'' - 4xy' + 6y = 0 \qquad (6)$$

This is done by part ①, ②.

By ①, ②. $\Rightarrow$ $y_1 = x^2$, $y_2 = x^3$

Step 2: Compute v_1, v_2

$$v_1 = \int \frac{-f\, y_2}{a W(y_1, y_2)}\, dx$$

Recall
$f = 4x^3$
$W(y_1, y_2) = x^4$

$\Rightarrow$

$$V_1 = \int \frac{-4x^3 \cdot x^3}{x^2 \cdot x^4}\, dx$$

$$= \int \frac{-4x^6}{x^6}\, dx = \int -4\, dx$$

$$= -4x + C$$

Choose $V_1 = -4x$

Similarly, $V_2 = \int \frac{f y_1}{a\, W(y_1, y_2)}\, dx$

$$= \int \frac{4x^3 \cdot x^2}{x^2 \cdot x^4}\, dx$$

$$= \int \frac{4}{x}\, dx = 4\ln|x| + C$$

Choose $V_2 = 4\ln|x|$

Step 3. obtain $y_p = V_1 y_1 + V_2 y_2$

$$y_p = (-4x) \cdot x^2 + 4\ln|x| \cdot x^3$$

$$= -4x^3 + 4\ln|x| \cdot x^3$$

is a particular soln to the D.E.