# Lecture 12

plan: §4.7 Variable-coefficient Equations

as opposed to "constant coefficient eqns"

$ay'' + by' + cy = f(x)$ → "constant coefficient"

E.g 1: $2y'' + 2y' + y = f(x)$

The coefficients are constants.

$a(x)y'' + b(x)y' + c(x)y = f(x)$ → "variable coefficient"

E.g 2: $x^2y'' - 4xy' + xy = f(x)$

The coefficients are functions of $x$.

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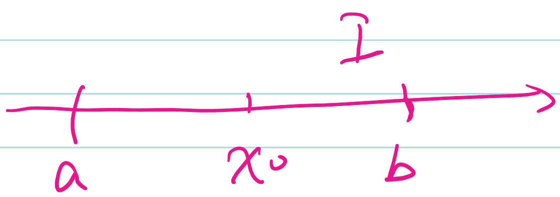
First we introduce two theorems about two special types of variable-coefficient equations.

Thm 1: Consider $y' + p(x) y = f(x)$

If $p, f$ are continuous on an interval $\underline{I} = (a, b)$ that contains $x_0$, then for any initial value $Y_0$, there exists a unique solution $y(x)$ defined on $I$ to the I.V.P.:

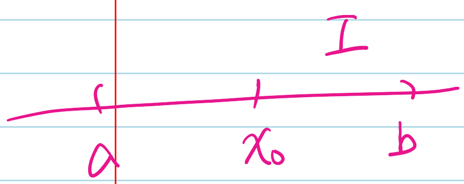
$$\begin{cases} y' + p(x) y = f(x) \\ y(x_0) = Y_0. \end{cases}$$



Thm 2. Consider $y'' + p(x)y' + q(x)y = f(x)$

If $p, q, f$ are continuous on an interval $I = (a, b)$ that containing $x_0$, then for any initial $Y_0, Y_1$, there exists a unique soln $y(x)$ defined on $I$ to the I.V.P:

$$\begin{cases} y'' + p(x)y' + q(x)y = f(x) \\ y(x_0) = Y_0, \quad y'(x_0) = Y_1 \end{cases}$$

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E.g. Does the I.V.P

$$\begin{cases} x^2 y'' + x y' + y = e^x \\ y(1) = 1, \quad y'(1) = \sqrt{2} \end{cases}$$

have a unique soln $y(x)$ on $I = (0, +\infty)$? Yes!

Divide the D.E by $x^2 \Rightarrow$

$$y'' + \underbrace{\frac{1}{x}}_{p(x)} y' + \underbrace{\frac{1}{x^2}}_{q(x)} y = \underbrace{\frac{e^x}{x^2}}_{f(x)}$$

Are $p, q, f$ continuous on $I = (0, +\infty)$?

$p(x)=\frac{1}{x}$
$q(x)$
$f(x)$

only bad at $x=0 \notin (0,+\infty)$

Note: In general, some I.V.P might not have a unique soln. Check the following E.g

E.g Consider

I.V.P $\begin{cases} y' = 3y^{\frac{2}{3}}, & (1) \\ y(2) = 0 \end{cases}$

The I.V.P has two distinct solns!

① check $y=0$ is a soln to the I.V.P

② check $y=(x-2)^3$ is also a soln to the I.V.P

A: ① Check $y=0$:

$$\text{LHS of } (1) = (0)' = 0$$

$$\text{RHS of } (1) = 3\cdot(0)^{2/3} = 0$$

$$\Rightarrow \text{LHS} = \text{RHS}$$

Moreover, $y(2)=0$ is satisfied!

why? "$y=0$" means "$y$ equals to $0$ at all points $x$"

② Check $y=(x-2)^3$

$$\text{LHS of (1)} = ((x-2)^3)' = 3(x-2)^2$$

$$\text{RHS of (1)} = 3y^{2/3} = 3((x-2)^3)^{2/3} = 3(x-2)^2$$

$\Rightarrow$ LHS = RHS

Moreover, $y(2) = (2-2)^3 = 0$ !

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Q: Can we find the two solns by ourselves in the above?

A: Yes. This is a seperable 1st order D.E

$$\frac{dy}{dx} = \underbrace{1}_{f(x)}\ \underbrace{3 \cdot y^{2/3}}_{g(y)}$$

Step 1: check whether $g(y)=0$ gives a soln.

$$g(y)=0 \Rightarrow y^{2/3}=0 \Rightarrow y=0$$

It is a soln, as we checked.

Step 2: Now assume $g(y)\neq 0$. Then separate the variables $\Rightarrow$

$$\frac{1}{3}\frac{dy}{y^{2/3}} = 1 \cdot dx$$

$$\Rightarrow \frac{1}{3} y^{-2/3}\, dy = dx$$

$$\Rightarrow \int \frac{1}{3} y^{-2/3}\, dy = \int 1 \cdot dx$$

$$\Rightarrow y^{1/3} = x + C$$

$$\Rightarrow y = (x+c)^3$$

$y(2) = 0 \Rightarrow 0 = (2+c)^3 \Rightarrow c = -2$

$x = 2$
$y = 0$

Hence $y = (x-2)^3$.

Next we discuss how to solve a special class of variable-coefficient eqns.

Recall in the end of lecture 11, we did the following

E.g:

Verify $y_1 = x^2$, $y_2 = x^3$ are solutions to the D.E

"$x^2 y'' - 4x y' + 4y = 0$"

The above D.E belongs to the class of so-called "Cauchy-Euler eqn".

<u>Def</u>$^n$: Cauchy-Euler eqn means the following 2nd order D.E

$$a x^2 y'' + b x y' + c y = f(x)$$

where $a, b, c \in \mathbb{R}$.

Remark: when we consider Cauchy-Euler eqn, most times we take $x > 0$.

E.g: "$3x^2 y'' + 11 x y' - 3y = 0$"

is a Cauchy-Euler eqn.

Q: How to find solns to

$$ax^2y'' + bxy' + cy = f(x), \quad x>0$$

A:

First consider the case $f=0 \Rightarrow$

$$ax^2y'' + bxy' + cy = 0 \qquad (2)$$

Hint: try the test function $y=x^r$, $r$ to be determined.

plug in $y=x^r$ (Note $y'=rx^{r-1}$, $y''=r(r-1)x^{r-2}$)

$$\Rightarrow \text{LHS of (2)} = ax^2r(r-1)x^{r-2} + bx \cdot rx^{r-1} + c \cdot x^r$$

$$= ar(r-1)x^r + b \cdot rx^r + cx^r$$

$$= (ar^2+(b-a)r+c)x^r$$

How to make it = RHS of (2) = 0 ?

Need $\quad ar^2+(b-a)r+c = 0$

It is called the associated characteristic eqn of the D.E.

E.g.: Find particular soln(s) to

$$3x^2y'' + 11xy' - 3y = 0 \quad \text{for } x > 0.$$

$(\Rightarrow a = 3, \quad b = 11, \quad c = -3)$

Use the above idea: try $y_p = x^r$

$\Rightarrow$ we need to solve

$$\text{“}ar^2 + (b-a)r + c = 0\text{”}$$

which is “$3r^2 + 8r - 3 = 0$”

$$\Leftrightarrow \quad (3r-1)(r+3) = 0$$

$\Rightarrow \quad r_1 = \frac{1}{3}, \quad r_2 = -3$ two distinct roots

Hence we get two particular solns:

$$y_1 = x^{\frac{1}{3}}, \quad y_2 = x^{-3}$$

In the above E.g, “$ar^2 + (b-a)r + c = 0$” has two distinct roots. What if it has only one repeated root or has complex roots $\alpha \pm \beta i$?

Here is the general algorithm:

Algorithm: find solns to

$$ax^2y'' + bxy' + cy = 0 \text{ for } x>0 \qquad (*)$$

Step 1: Solve the quadratic eqn ← char.

$$ar^2 + (b-a)r + c = 0 \qquad (3)$$

Step 2:

(I) If (3) has two distinct real roots $r_1, r_2$, ($\Delta>0$)

then we have two linearly independent solns:

$$y_1 = x^{r_1}, \quad y_2 = x^{r_2}$$

(II) If (3) has only one repeated root $r_0$, ($\Delta=0$)

then we have two li ly independent solns:

$$y_1 = x^{r_0}, \quad y_2 = x^{r_0}\ln x$$

(III) If (3) has complex roots $\alpha \pm \beta i$, ($\Delta<0$)

$\beta \neq 0$, then we have two linearly independent solns:

$$y_1 = x^{\alpha}\cos(\beta \ln x), \quad y_2 = x^{\alpha}\sin(\beta \ln x)$$

In any of the cases (I), (II), (III), the general soln to (*) is

$$y = c_1 y_1 + c_2 y_2, \quad c_1, c_2 \in \mathbb{R}$$

We will skip the proof for the above algorithm.

But we briefly discuss case (II).

Q: In Case (II), why $y_2 = x^{r_0} \ln x$ is a soln ?

A: In case (II), $r_0$ is the repeated root of

$$\text{"}\ \underbrace{a}_{A} r^2 + \underbrace{(b-a)}_{B} r + \underbrace{c}_{C} = 0\ \text{"}$$

We thus have

$$B^2 - 4AC = \Delta = (b-a)^2 - 4ac = 0$$

and by quadratic formula,

$$r_0 = \frac{-(b-a) \pm \sqrt{\Delta}}{2a} = \frac{b-a}{-2a}$$

$$\Rightarrow \quad -2a r_0 = b-a \Rightarrow 2a r_0 + b - a = 0$$

We next plug in $y_2 = x^{r_0}\ln x$ to

$$ax^2y'' + bxy' + cy = 0$$

Note $y_2' = r_0 x^{r_0-1}\ln x + x^{r_0-1}$

$$y_2'' = r_0(r_0-1)x^{r_0-2}\ln x + (2r_0-1)x^{r_0-2}$$

$\Rightarrow$

$$ax^2y_2'' + bxy_2' + cy_2$$

$$= ar_0(r_0-1)x^{r_0}\ln x + a(2r_0-1)x^{r_0} + br_0x^{r_0}\ln x$$

$$+ bx^{r_0} + cx^{r_0}\ln x$$

$$= \underbrace{(ar_0^2+(b-a)r_0+c)}_{=0}x^{r_0}\ln x + \underbrace{(2ar_0+b-a)}_{=0}x^{r_0}$$

$$= 0$$

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Now by the above algorithm, we can solve

$$ax^2y'' + bxy' + cy = 0 \qquad (*)$$

Can we solve the general eqn:

$$ax^2y'' + bxy' + cy = f(x) \qquad (4)\ ?$$

Yes! This was indeed already done by Lecture 11, variation of parameters.

Suppose $y_1, y_2$ are two linearly independent solns. to $(*)$, then we have a particular soln to (4):

Warning: $a(x) =$ coeff. of $y''$ $= ax^2$

$$y_p = v_1 y_1 + v_2 y_2$$

$$v_1 = \int \frac{-f y_2}{a(x) W(y_1, y_2)} dx, \quad v_2 = \int \frac{f y_1}{a(x) W(y_1, y_2)} dx$$

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E.g Find the general soln to

$$x^2 y'' + 5 x y' + 5y = 0 \text{ for } x > 0$$

$a=1$ $b=5$ $c=5$

Step 1: Solve $ar^2 + (b-a)r + c = 0$

$$\Rightarrow \quad r^2 + 4r + 5 = 0$$

$\alpha \pm \beta i$

$$\Rightarrow \quad r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2 \cdot 1} = -2 \pm i$$

$\Rightarrow \begin{cases} \alpha = -2 \\ \beta = 1 \end{cases}$ Hence we have complex roots:

$$y_1 = x^{-2}\cos(\ln x), \quad y_2 = x^{-2}\sin(\ln x)$$

General soln:

$$c_1 y_1 + c_2 y_2 = c_1 x^{-2}\cos(\ln x) + c_2 x^{-2}\sin(\ln x)$$

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E.g: Solve the I.V.P

we can assume that x is close to $x_0 = 1$. Thus $x > 0$

$$\begin{cases} y'' + \frac{1}{x} y' = 0 \\ y(1) = 1, \; y'(1) = 2 \end{cases}$$

A: Note $y'' + \frac{1}{x} y' = 0$ is NOT a Cauchy-Euler eqn.

$ax^2y'' + bxy' + cy = 0$

But we can multiply it by $x^2$: $\Rightarrow$

$$x^2y'' + xy' = 0$$

$1 \cdot x^2y'' + 1 \cdot xy' + 0 \cdot y = 0$

$a = 1, \quad b = 1, \quad c = 0$

Step 1: Solve $ar^2 + (b-a)r + c = 0$

$$\Rightarrow \quad r^2 + 0 = 0$$

$$\text{or} \quad r^2 = 0$$

$$\Rightarrow \quad r_0 = 0 \quad \text{, repeated root}$$

Step 2: Since we have only one repeated root: $r_0 = 0 \Rightarrow$

$$y_1 = x^{r_0} = x^0 = 1$$

$$y_2 = x^{r_0} \ln x = \ln x$$

And the general soln

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 + c_2 \ln x$$

Step 3: plug in $y(1) = 1$, $y'(1) = 2$

$x=1$
$y=1$ $\leftarrow$ $y(1) = 1 \Rightarrow 1 = c_1 + c_2 \overset{0}{\cancel{\ln 1}}$

$$\Rightarrow \quad C_1 = 1$$

To use $y'(1) = 2$, note

$$y' = \underline{C_2 \frac{1}{x}}$$

$y'(1) = 2 \quad \Rightarrow \quad 2 = C_2 \cdot 1$

$x=1$

$y'=2 \qquad \Rightarrow \quad C_2 = 2$

$$\Rightarrow \quad y = 1 + 2\ln x$$

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E.g. Find a particular soln to

$$x^2 y'' + x y' = \underbrace{1}_{f(x)} \quad \underline{\underline{\text{for } x > 0}} \quad (5)$$

A: Step 1: Find two L.I solns to the <u>homogeneous eqn</u>:

$$x^2 y'' + x y' = 0$$

We already did in the above: $y_1 = 1$, $y_2 = \ln x$

Step 2: Use variation of parameters to find a particular soln to (5):

$a(x)$ = coeff. of $y''$

$$y_p = v_1 y_1 + v_2 y_2$$

Here $v_1 = \int \frac{-f y_2}{a(x) W(y_1, y_2)} dx$

Note

$a(x) = x^2$

$f = 1$

$$v_2 = \int \frac{f y_1}{a(x) W(y_1, y_2)} dx ,$$

and $W(y_1, y_2)$

$$= y_1 y_2' - y_2 y_1'$$

$$= 1 (\ln x)' - \ln x (1)' = \frac{1}{x}$$

$$\Rightarrow v_1 = \int \frac{-\underbrace{1}_{f} \cdot \underbrace{\ln x}_{y_2}}{\underbrace{x^2}_{a(x)} \cdot \underbrace{\frac{1}{x}}_{W}} dx = \int -\frac{\ln x}{x}$$

$u = \ln x$

$$= -\frac{1}{2} (\ln x)^2 + C$$

u-sub

Choose $v_1 = -\frac{1}{2} (\ln x)^2$

$$V_2 = \int \frac{1 \cdot 1}{x^2 \cdot \frac{1}{x}} dx = \int \frac{1}{x} dx = \ln x + C$$

choose $v_2 = \ln x$

$\Rightarrow$ $y_p = v_1 y_1 + v_2 y_2$

$$= -\frac{1}{2}(\ln x)^2 \cdot 1 + (\ln x)(\ln x)$$

$$= \frac{1}{2}(\ln x)^2$$