Lecture 15

plan: • Recap §7.2, §7.3

• Discuss §7.4 Inverse Laplace transform

• §7.2: Important Table of Laplace transform

$f(t)$	$\mathcal{L}\{f\}(s)$	Region of existence
c (a constant)	$\frac{c}{s}$	$s>0$
t^n, $n>0$ integer	$\frac{n!}{s^{n+1}}$	$s>0$
e^{kt}, $k\in\mathbb{R}$	$\frac{1}{s-k}$	$s>k$
$\sin bt$, $b\in\mathbb{R}$	$\frac{b}{s^2+b^2}$	$s>0$
$\cos bt$, $b\in\mathbb{R}$	$\frac{s}{s^2+b^2}$	$s>0$
$e^{at}\sin(bt)$, $a,b\in\mathbb{R}$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos(bt)$, $a,b\in\mathbb{R}$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$

§7.3.

Thm 1 (Translation):

If $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$, then

$$\mathcal{L}\{e^{\beta t} f(t)\}(s) = \mathcal{L}\{f\}(s-\beta)$$

for $s > \alpha + \beta$.

Thm 2 (Derivative).

Let f be continuous on $[0, \infty)$ and assume f' exists. Assume f' is piecewise continuous on $[0, \infty)$. with both of exponential order α. Then for $s > \alpha$,

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

Thm 3. (Integral) The following holds whenever the two Laplace transforms:

$$\mathcal{L}\left\{\int_0^t f(x)dx\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s)$$

Thm 4: (Higher-order derivatives)

Assume $f, f', \cdots, f^{(n-1)}, f^{(n)}$ are all piecewise continuous on $[0, \infty)$, and are of exponential order α. Then for $s > \alpha$.

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Thm 5: (Multiply of t^n).

Assume $F(s) = \mathcal{L}\{f\}(s)$ for $s > \alpha$.

Then for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s),$$

$$n \in \mathbb{Z},\ n \geq 1$$

This lecture:

In the past lectures, for a given f, we try to compute $F = \mathcal{L}\{f\}$. Now, can we do the converse: Given F, can we find f such that

$$\mathcal{L}\{f\} = F\ ?$$

Notation: If $\mathcal{L}\{f\} = F$, then we write

$$f = \mathcal{L}^{-1}\{F\} \qquad f \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F$$

E.g: Given $F(s) = \frac{10}{s^3}$, find $\mathcal{L}^{-1}\{F\}$.

(means, need to find f s.t $\mathcal{L}\{f\} = F$)

A: Recall from the table,

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}, \qquad s>0$$

Let $n=2$, $\Rightarrow$ $\mathcal{L}\{t^2\}(s) = \frac{2}{s^3}$

$$\Rightarrow \mathcal{L}\{\overset{f}{5t^2}\}(s) = 5\cdot\frac{2}{s^3} = \overset{F}{\frac{10}{s^3}}$$

$$= 5\mathcal{L}\{t^2\} =$$

Hence $\mathcal{L}^{-1}\{\frac{10}{s^3}\} = 5t^2$

$$\mathcal{L}^{-1}\{5\cdot\frac{2}{s^3}\} = 5\cdot\underbrace{\mathcal{L}^{-1}\{\frac{2}{s^3}\}}_{t^2}$$

In general, we have

Thm. (Linearity of $\mathcal{L}^{-1}$)

For $c_1, c_2 \in \mathbb{R}$.

$$\mathcal{L}^{-1}\{c_1F_1 + c_2F_2\} = c_1\mathcal{L}^{-1}\{F_1\} + c_2\mathcal{L}^{-1}\{F_2\}$$

$$\overset{c_1}{5}\cdot\frac{2}{s^3}+\overset{c_2}{1}\cdot\frac{s-1}{s^2-2s+5}$$

E.g. Let $F(s)=\frac{10}{s^3}+\frac{s-1}{s^2-2s+5}$

Find $\mathcal{L}^{-1}\{F\}$.

$(s^2-2s+5)=s^2-2s+1+4$

A:

$$\mathcal{L}^{-1}\{F\}(t)=5\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}+\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}$$

$$=5\cdot t^2+\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}$$

$$=5t^2+e^t\cos(2t)$$

Recall for $s>a$

$$\mathcal{L}\{e^{at}\cos(bt)\}(s)=\frac{s-a}{(s-a)^2+b^2}$$

$\Rightarrow$ Let $a=1,\ b=2$

$$\mathcal{L}\{e^t\cos(2t)\}(s)=\frac{s-1}{(s-1)^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}=e^t\cos(2t)$$

In general, we have an algorithm to find $\mathcal{L}^{-1}\{F\}$

for $F=\frac{P}{Q}$, with $\deg P<\deg Q$

P, Q are polynomials

P.F.D

For that, we first recall partial fraction decomposition:

Let $F = \frac{P}{Q}$, with $\deg P < \deg Q$

Algorithm for P.F.D:

(distinct roots)

- If $Q = (x-\alpha_1)\cdots(x-\alpha_n)$, where $\alpha_1, \ldots, \alpha_n$ are n distinct roots of $Q=0$. Then

$$\frac{P}{Q} = \frac{P}{(x-\alpha_1)\cdots(x-\alpha_n)}$$

$$= \frac{C_1}{(x-\alpha_1)} + \cdots + \frac{C_n}{(x-\alpha_n)} \qquad (*)$$

($C_1, \cdots, C_n$ are to be determined)

repeated roots

- If $Q=0$ has repeated root a, that is, Q has a factor $(x-a)^n$, then the RHS of $(*)$ will have

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

($A_1, \cdots, A_n$ are to be determined) no real roots

- (Quadratic factor) If Q has an N-th power of Irreducible quadratic factor (x^2+bx+c) (that is, Q has a factor $(x^2+bx+c)^N$)

then RHS of $(*)$ will have

$$\frac{A_1x+B_1}{(x^2+bx+c)}+\frac{A_2x+B_2}{(x^2+bx+c)^2}+\cdots+\frac{A_Nx+B_N}{(x^2+bx+c)^N}.$$

($A_1, \cdots, A_N, B_1, \cdots, B_N$ are to be determined)

What is an "irreducible" factor?

"x^2+bx+c" is called irreducible if

$(x^2+bx+c) \neq (x+a_1)(x+a_2)$ for any $a_1, a_2 \in \mathbb{R}$

"x^2+bx+c" is irreducible

$\Longleftrightarrow$

$\Delta = b^2-4c<0$

$\Longleftrightarrow$ "$x^2+bx+c=0$" has no real roots.

E.g 1: Let $F(s) = \dfrac{s^2+9s+2}{(s-1)^2(s+3)}$. Find its partial fractional decomposition. (P.F.D)

P

Q

A: Since Q has $(s+3)$, $\Rightarrow$ P.F.D has

$$\frac{C}{s+3}$$

Since Q has $(s-1)^2 \Rightarrow$ P.F.D has

$$\frac{A}{s-1} + \frac{B}{(s-1)^2}$$

Hence $$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3} \quad (1)$$

(A, B, C are to be determined)

Q: How to find the values of A, B, C ?

A: Multiply (1) by $\overset{Q}{\overset{\shortparallel}{(s-1)^2(s+3)}} \Rightarrow$

$$s^2+9s+2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2 \quad (2)$$

Two ways to find A, B, C

Way 1: Mutiply out and compute RHS of (2).

$\Rightarrow$ RHS of (2) $= (A+C)s^2 + (2A+B-2C)s$
$+ (-3A+3B+C)$

LHS of (2) $= s^2+9s+2$

Compare: LHS v.s. RHS

s^2-term: $1 = A+C$

s-term: $9 = (2A+B-2C)$

Constant $2 = -3A+3B+C$

$$\Longrightarrow \begin{cases} A=2 \\ B=3 \\ C=-1 \end{cases}$$

Way 2: plug in roots of Q into (2):

$s^2+9s+2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2 \quad (2)$

Let $s=1$ in (2) $\Longrightarrow$

$12 = B\cdot 4 \Rightarrow B=3$

Let $s=-3$ in (2) $\Longrightarrow$

$9-27+2 = C\cdot(-3-1)^2$

$$\Rightarrow \quad -16 = 16C \Rightarrow C = -1$$

Then plug some other "simple" number for s.

different from the roots of Q

$|s|$ is small

Let $s=0 \Rightarrow$

$$0^2 + 9\cdot 0 + 2 = A(-1)(3) + 3\underset{\substack{\parallel \\ 3}}{B} + \underset{\substack{\parallel \\ -1}}{C}$$

$$\Rightarrow \quad A = 2$$

Hence

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{2}{s-1} + \frac{3}{(s-1)^2} + \frac{-1}{s+3}$$

E.g 2: Find the P.F.D of $\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}$ $\leftarrow P$, $\leftarrow Q$

A: Since Q has the factor (s^2-2s+5).

$\Delta = (-2)^2 - 4\cdot 5 = -16 < 0$

Yes! irreducible

$\Rightarrow$ the P.F.D has $\dfrac{As+B}{(s^2-2s+5)}$

Since Q has the factor $(s+1)$

$\Rightarrow$ the P.F.D has $\dfrac{C}{s+1}$.

Hence $$\frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{As+B}{(s^2-2s+5)} + \frac{C}{s+1} \quad (3)$$

Next we find values of A, B, C:

Multiply (3) by $\underbrace{(s^2-2s+5)(s+1)}_{Q}$

$$2s^2+10s = (As+B)(s+1) + C(s^2-2s+5) \quad (5)$$

Way 1: Multiply out RHS of (5) and compare both sides of (5)

Exercise.

Way 2: plug $s=-1$ into (5)

$$\Rightarrow \quad 2-10 = C(1+2+5)$$

$$\Rightarrow \quad C = -1$$

plug some "simply" other values of s

$s=0 \Rightarrow$

$$0 = B + 5C \Rightarrow B = 5$$

$s = 1 \Rightarrow$

$$2+10 = (A+B)\cdot 2 + C(1-2+5)$$

$$\Rightarrow 12 = 2(A+B) + 4C$$

$$\Rightarrow A = 3$$

$$\Rightarrow \frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{3s+5}{s^2-2s+5} + \frac{-1}{s+1}$$

More examples about P.F.D:

E.g 3: Determine the format of the P.F.D (no need to compute the constant).

① $\dfrac{2s^2+s+1}{s^4+s^3} \leftarrow Q$

② $\dfrac{3s^2}{(2s+5)^3}$

A: ① $\dfrac{2s^2+s+1}{s^4+s^3} = \dfrac{2s^2+s+1}{\underbrace{s^3}_{(s-0)^3}(s+1)}$

$$= \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s^3} + \frac{B}{s+1}$$

If Q has $(s-a)^n$ $\dashrightarrow$ P.F.D of $\frac{P}{Q}$ has $\dfrac{A_1}{(s-a)} + \dfrac{A_2}{(s-a)^2} + \cdots + \dfrac{A_n}{(s-a)^n}$

② $\frac{3s^2}{(2s+5)^3} = \frac{3s^2}{2^3(s+\frac{5}{2})^3}$

$= \frac{1}{2^3} \frac{3s^2}{(s+\frac{5}{2})^3}$

$= \frac{1}{8}\left(\frac{A_1}{s+\frac{5}{2}} + \frac{A_2}{(s+\frac{5}{2})^2} + \frac{A_3}{(s+\frac{5}{2})^3} \right)$

$= \frac{B_1}{s+\frac{5}{2}} + \frac{B_2}{(s+\frac{5}{2})^2} + \frac{B_3}{(s+\frac{5}{2})^3}$

$\begin{cases} B_1 = \frac{1}{8}A_1 \\ \vdots \\ B_3 = \frac{1}{8}A_3 \end{cases}$

Now we discuss how to find $\mathcal{L}^{-1}\{f\}$ for $f = \frac{P}{Q}$

where P, Q are polynomials with $\deg P < \deg Q$.

E.g 4: Find $\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$

$\uparrow \frac{P}{Q}$

A:

step 1: Find the P.F.D for $\frac{P}{Q}$

We already did this in E.g 2.

$\frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{3s+5}{(s^2-2s+5)} + \frac{-1}{s+1}$.

Step 2: Use the table in §7.2 and Thms (Thm 1-5) in §7.3 if needed to find $\mathcal{L}^{-1}\{f\}$.

Note

$$\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3s+5}{(s^2-2s+5)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\downarrow \qquad\qquad \downarrow$$

$$? \qquad\qquad -e^{-t}$$

Recall for $k \in \mathbb{R}$

$$\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$$

Let $k=-1 \Rightarrow$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

To find $\mathcal{L}^{-1}\left\{\frac{3s+5}{(s^2-2s+5)}\right\}$

when the denominator is quadratic, write it into the form

$$(s-a)^2 + b^2$$

by completing the square.

$$\begin{aligned}(s^2-2s+5) &= s^2-2s+1+4 \\ &= (s-1)^2+4 \\ &= (s-1)^2+2^2\end{aligned} \Rightarrow \begin{aligned} a&=1 \\ b&=2\end{aligned}$$

Then write the numerator into the form $A(s-a)+B \Rightarrow A(s-1)+B$

$$3s+5 = 3(s-1)+8$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{3s+5}{s^2-2s+5}\right\} \overset{?}{=} \mathcal{L}^{-1}\left\{\frac{3(s-1)+8}{(s-1)^2+2^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3(s-1)}{(s-1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{8}{(s-1)^2+2^2}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}$$

$= 3e^{t}\cos(2t) + 4e^{t}\sin(2t)$

Recall

$$\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$$

$\Rightarrow$

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos(bt)$$

Let $a=1$, $b=2$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} = e^{t}\cos(2t)$$

Recall

$$\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin(bt)$$

Let $a=1$, $b=2$

$\Rightarrow$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\} = e^{t}\sin(2t)$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}(t)$$

$$= 3e^{t}\cos(2t) + 4e^{t}\sin(2t) - e^{-t}$$

E.g 5. Find $\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$

A: Step 1: Find the P.F.D of $\frac{P}{Q}$.

We already did this in E.g 1

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{2}{s-1} + \frac{3}{(s-1)^2} + \frac{-1}{s+3}$$

Step 2. Use the table and Thms in §7.2 and §7.3

$$\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$$

$$= 2\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}}_{e^t} + 3\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}}_{te^t} - \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}}_{e^{-3t}}$$

Recall

$\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$

$\Rightarrow$

$\mathcal{L}^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

Q: How to compute $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$?

Let $k=1 \Rightarrow$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}$$

Let $k=-3 \Rightarrow$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

A: Since

$$\mathcal{L}\{e^{t}\}(s) = \frac{1}{s-1} \leftarrow F$$

$$\Rightarrow \mathcal{L}\{te^{t}\}(s) = (-1)\frac{dF}{ds}$$

Thm 5 in §7.3

$$= -\left(\frac{1}{s-1}\right)'$$

$$= \frac{1}{(s-1)^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t) = te^{t}$$

Hence $\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}(t)$

$$= 2e^{t} + 3te^{t} - e^{-3t}$$

In the above, we used the formula that is NOT in the table: $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t) = te^{t}$

In general, we have the following three formulas that are not in the table and can be useful:

(1) $\mathcal{L}^{-1}\left\{\frac{1}{(s-k)^2}\right\}(t) = te^{kt}$

(2) $\mathcal{L}^{-1}\left\{\frac{2bs}{(s^2+b^2)^2}\right\}(t) = t\sin(bt)$

(3) $\mathcal{L}^{-1}\left\{\frac{s^2-b^2}{(s^2+b^2)^2}\right\}(t) = t\cos(bt)$

E.x prove them by using the table in §7.2 and Thm 5 in §7.3.