Lecture 20

plan: §7.9 The Dirac Delta

Def^n: The Dirac delta function $\delta(t)$ → delta

is a map $\delta: \underbrace{\mathbb{R}}_{X} \to \underbrace{\mathbb{R} \cup \{\infty\}}_{Y}$ that satisfies

① $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$

means "$+\infty$"

② $$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

for any function $f: \mathbb{R} \to \mathbb{R}$ that is continuous on some interval containing 0.

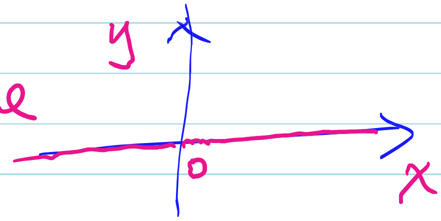
Remark :

- Regard ①. δ is NOT a typical function as we learned in Calculus. In Calculus, "functions" are maps from a subset

$S \subseteq \mathbb{R} \to \mathbb{R}$.

But δ takes values in $\mathbb{R} \cup \{\infty\}$.

Note: we cannot plot "∞" on x-y plane



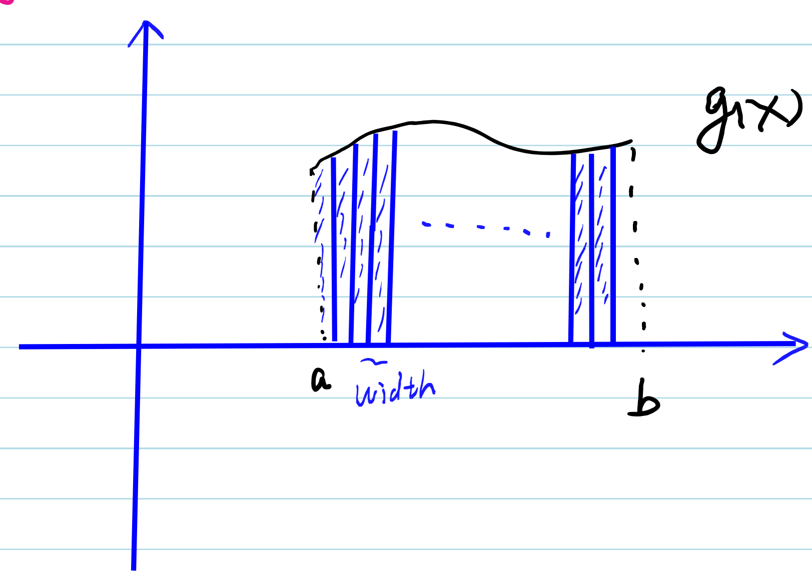
· Regarding ②, the integral

$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dt$ cannot be understood

in the calculus way.

In calculus, we understand an integral

$\int_a^b g(x)\,dx$ in terms of "Riemann sum".

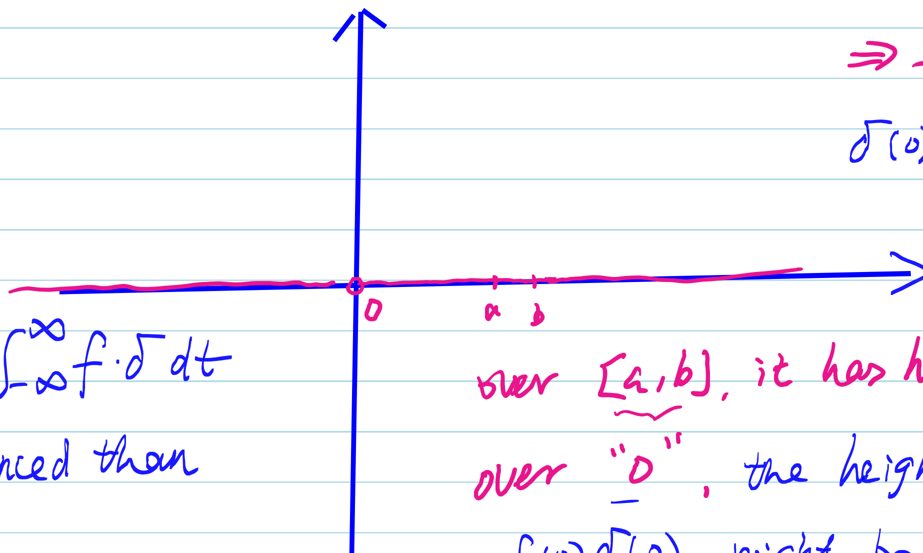


How about $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt$?

$t \neq 0$

$\Rightarrow f(t)\delta(t) = 0$

$\delta(0) = \infty$



The def^n of $\int_{-\infty}^{\infty} f \cdot \delta \, dt$ is more advanced than Calculus

over $[a, b]$, it has height 0

over "0", the height is $f(0)\delta(0)$, might be ∞

width · height $= 0 \cdot \infty \rightarrow$ No definitive answer.

Q: why $\delta(t)$ is useful?

A: Indeed, $\delta(t)$ can rise in a practical/physics problem.

E.g.: density of the universe

part of the universe:

black hole

vacuum   vacuum

density: 0 ∞ 0

Another example will be given at the end of the lecture.

Properties:

Assume $f(t)$ is continuous on $(-\infty, \infty)$

(1). $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$

(2). The Laplace transform for $a \geq 0$,

☆ $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$.

Intuitive pf of (1):

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt$$

Use u-sub: $u = t-a \Rightarrow du = dt,\ t = u+a$

$\Rightarrow$ the above $= \int_{-\infty}^{\infty} f(u+a)\,\delta(u)\,du$

Let $g(u) = f(u+a)$

$\Downarrow$ let $u=0$

$g(0) = f(a)$

$$= \int_{-\infty}^{\infty} g(u)\,\delta(u)\,du$$

$$= g(0)$$

$$= f(a).$$

Pf of (2): By the defn of Laplace transform,

$$\mathcal{L}\{\underbrace{\delta(t-a)}_{h(t)}\}(s)$$

$$= \int_0^\infty e^{-st}\, \underbrace{\delta(t-a)}_{h(t)}\, dt$$

Add $\int_{-\infty}^{0} e^{-st}\delta(t-a)\,dt = 0$

$$= \int_{-\infty}^\infty \underbrace{e^{-st}}_{f(t)}\, \delta(t-a)\, dt$$

$$= f(a) = e^{-sa} = e^{-as}$$

Hint:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

$x = t-a \Longrightarrow$

$$\delta(t-a) = \begin{cases} 0 & \text{if } t \neq a \\ \infty & \text{if } t = a \end{cases}$$

$\Rightarrow$ when $t < a \Rightarrow \delta(t-a) = 0$

when $t < 0 \Longrightarrow t < a$

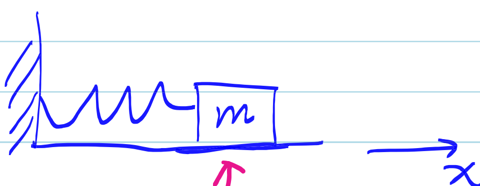
$\Rightarrow \delta(t-a) = 0$

$\Rightarrow e^{-st}\delta(t-a) = 0$ when $t < 0$

$\Rightarrow \int_{-\infty}^{0} e^{-st}\delta(t-a)\,dt = 0$

E.g.:

A mass attached to a spring



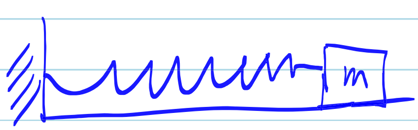
rest position: $x = 0$

$x(t)$: the position of m at time t.

Assume:

At time $t=0$:

m is released from $x=1$



$x(0) = 1$

At time $t = \pi$,

m is struck by a hammer exerting an impulse on m.

Then the system is governed by the following I.V.P

$$\begin{cases} x'' + 9x = 3\delta(t-\pi) & (*) \\ x(0) = 1,\ x'(0) = 0. \end{cases}$$

Solve the above I.V.P.

A: Step 1: Apply Laplace transform $\mathcal{L}$ to the D.E $(*)$

$$\mathcal{L}\{x'' + 9x\} = \mathcal{L}\{3\delta(t-\pi)\} \quad (1)$$

write $X = \mathcal{L}\{x\}$.

$$\Rightarrow \mathcal{L}\{x''\} = s^2\mathcal{L}\{x\} - s\,\cancel{x(0)}^{1} - \cancel{x'(0)}^{0}$$

$$= s^2 X - s$$

Thm 4 §7.3

$$\Rightarrow \text{LHS of } (1) = \mathcal{L}\{x''\} + 9\mathcal{L}\{x\}$$

$$= s^2 X - s + 9X$$

$$\text{RHS of } (1) = 3\mathcal{L}\{\delta(t-\pi)\}(s)$$

$$= 3e^{-\pi s}$$

$$\Rightarrow s^2 X - s + 9X = 3e^{-\pi s}$$

$$\Rightarrow \mathcal{L}\{x\} = X = \frac{s}{s^2+9} + e^{-\pi s}\frac{3}{s^2+9}$$

$$\Rightarrow x = \mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3\cdot e^{-\pi s}}{s^2+9}\right\}$$

$$= \underset{\cos 3t}{\underset{\parallel}{}} \qquad \underset{?}{\underset{\parallel}{}}$$

Table in §7.2

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin bt.$$

Recall §7.6, Lecture 17

if $F(s) = \mathcal{L}\{f\}(s)$

then $\mathcal{L}^{-1}\{e^{-cs}F(s)\}$

$= f(t-c)\,u(t-c)$

u: step function

Then $\mathcal{L}^{-1}\left\{\frac{3\cdot e^{-\pi s}}{s^2+9}\right\}$

$$= \mathcal{L}^{-1}\left\{\underset{c=\pi}{e^{-\pi s}} \cdot \underset{F(s)}{\frac{3}{s^2+9}}\right\}$$

We need to find f s.t $F = \mathcal{L}\{f\}(s)$

Note $\frac{3}{s^2+9} = \mathcal{L}\{\sin 3t\}(s)$

$$\Rightarrow f(t) = \sin 3t$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{3e^{-\pi s}}{s^2+9}\right\} = f(t-\pi)\,u(t-\pi)$$

$$= \sin 3(t-\pi)\,u(t-\pi)$$

Hence

$$x(t) = \cos 3t + \sin 3(t-\pi)\, u(t-\pi).$$