Lecture 22

Plan:

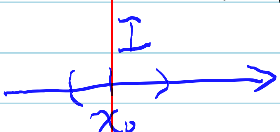
- finish §8.2 power series
- discuss part of §8.3 power series solution to D.Es

We first discuss the remaining part of §8.2

Def^n: (Analytic function)

A function $f(x)$ is said to be analytic at $x_0 \in \mathbb{R}$ if

in an open interval I containing x_0,



$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

called Taylor series of f at x_0

properties: Let f, g be analytic at x_0. Then

① For any $k \in \mathbb{R}$, kf is analytic at x_0.

② $f+g$, $f-g$ are analytic at x_0

③ fg is analytic at x_0.

④ $\frac{f}{g}$ is analytic at x_0 "providing that $g(x_0) \neq 0$".

☆ Thm: (Taylor's Thm)

If $f(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$ on an open interval I containing x_0, then $a_n = \frac{f^{(n)}(x_0)}{n!}$.

Consequently, $$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Important Taylor series (about $x_0=0$) we need to know:

① $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+\cdots, \quad |x|<1$

② $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2}+\cdots, \quad x\in\mathbb{R}$

③ $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad x\in\mathbb{R}$

④ $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \quad x\in\mathbb{R}$

⑤ $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots, \quad |x|<1$

E.x: prove ⑤ using ①. Hint: First find Taylor series of $\frac{1}{1+x}$, and then integrate.

§8.3 Power series solutions to D.Es

Def^n: Consider the D.E

$$y' + b(x)\,y = 0 \qquad (1)$$

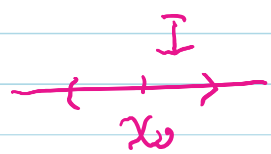
We say $x = x_0$ is an ordinary pt of the D.E (1) if $b(x)$ is analytic at x_0. Otherwise, it is a singular pt of the D.E (1).

Def^n: Consider the D.E

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad (2)$$

We say $x = x_0$ is an ordinary pt of the D.E (2) if $p(x)$ and $q(x)$ are analytic at x_0. Otherwise, it is called a singular pt of the D.E (2).

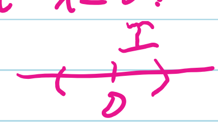
Remark: On a small open interval I centered at an ordinary pt x_0, we can expect the D.E to have a power series sol^n.



E.g.: Find a power series solution $\underbrace{\text{about}}$ $x = 0$ to

$$y' + \underbrace{2x}_{b(x)}\,y = 0$$

means on a small interval centered at $x = 0$.



Remark: $x = 0$ is an ordinary pt for the D.E .

A: We try a power series solution.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad x_0 = 0$$

a_n is to be determined.

$$\Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Recall
if $f(x) = \sum_{n=0}^{\infty} a_n x^n$
$\Rightarrow$
$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$
$= \sum_{n=0}^{\infty} n a_n x^{n-1}$

$n=0$, $n a_n x^{n-1} = 0$

Then "$y' + 2xy = 0$" $\Rightarrow$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

Next we shift the summation indices to make the general term be x^k.

① $\sum_{n=1}^{\infty} n a_n x^{n-1}$ $\xrightarrow{\text{Let } k=n-1 \;\Rightarrow\; n=k+1}$ $\sum_{k+1=1}^{\infty} (k+1) a_{k+1} x^k$

$$= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k$$

(2) $\sum_{n=0}^{\infty} 2a_n x^{n+1} \xrightarrow[\substack{\text{let } k=n+1 \\ \Rightarrow n=k-1}]{} \sum_{k-1=0}^{\infty} 2a_{k-1}x^k$

$$= \sum_{k=1}^{\infty} 2a_{k-1}x^k$$

Hence the equation becomes

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=1}^{\infty} 2a_{k-1}x^k = 0$$

$$\underbrace{a_1}_{k=0} + \sum_{k=1}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=1}^{\infty} 2a_{k-1}x^k = 0$$

$$\Rightarrow \quad a_1 + \sum_{k=1}^{\infty}\left((k+1)a_{k+1} + 2a_{k-1}\right)x^k = 0$$

$\Rightarrow$ All coefficients of x^k for $k \geq 0$ on LHS must equal to 0.

$1 = x^0$: $\quad a_1 = 0$

$x^k, k \geq 1$: $\quad (k+1)a_{k+1} + 2a_{k-1} = 0$

$$\Rightarrow \quad \underbrace{a_{k+1} = -\frac{2}{k+1}a_{k-1}}_{\text{a recurrence relation}} \quad \text{for } k \geq 1$$

Let's write this up one by one:

$$k=1 \Rightarrow a_2 = -\frac{2}{2}a_0 = -a_0$$

$$k=2 \Rightarrow a_3 = -\frac{2}{3}a_1 = 0$$

$$k=3 \Rightarrow a_4 = \frac{-2}{4}a_2 = -\frac{1}{2}a_2 = \frac{1}{2!}a_0$$

$$k=4 \Rightarrow a_5 = -\frac{2}{5}a_3 = 0$$

$$\cancel{k=6} \Rightarrow a_6 = \frac{-2}{6}a_4 = -\frac{1}{3!}a_0$$

should be $k=5$

$$\cancel{k=7} \Rightarrow a_7 = \frac{-2}{7}a_5 = 0$$

should be $k=6$

$$\cancel{k=8} \Rightarrow a_8 = \frac{-2}{8}a_6 = +\frac{1}{4!}a_0$$

should be $k=7$

In general, we have

To prove it, we need to use proof by induction (Math 109)

$$a_{2n+1} = 0 \quad \text{for all } n \geq 0$$

$$a_{2n} = \frac{(-1)^n}{n!}a_0 \quad \text{for } n \geq 0.$$

Thus NOT required

Now $\quad y = \sum_{k=0}^{\infty} a_k x^k$

$$= \cancel{\sum_{k \text{ odd}} a_k x^k} + \sum_{k \text{ even}} a_k x^k$$

0

$$= \sum_{k \text{ even}} a_k x^k$$

Let $k = 2n$, $n \geq 0$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a_0 x^{2n}$$

$$= a_0 \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \right), \quad a_0 \in \mathbb{R}$$

This is the power series solution to '$y' + 2xy = 0$'.

Remark: We can indeed solve '$y'+2xy=0$' by using Separation of variables.

$$y' + 2xy = 0$$

$$\Rightarrow \frac{dy}{dx} = -2xy \qquad ① \ y = 0 \text{ is a soln}$$

$$② \ y \neq 0 \Rightarrow \frac{dy}{y} = -2x\,dx$$

$$\Rightarrow \int \frac{dy}{y} = \int -2x\,dx$$

$$\Rightarrow \ln|y| = -x^2 + C$$

$$\Rightarrow y = \pm e^C e^{-x^2}$$

$$\Rightarrow y = A e^{-x^2}$$

$A = \pm e^C$, and $A = 0$ gives $y = 0$ in ①

Q: Are the two solutions the same?

$$\begin{cases} y = a_0\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\right), & a_0 \in \mathbb{R} \\ y = A e^{-x^2}, & A \in \mathbb{R} \end{cases}$$

A: Yes!

What's the Taylor expansion of e^{-x^2} at $x=0$?

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad t \in \mathbb{R}$$

$\Longrightarrow$ Let $t = -x^2$

Substitution

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}, \quad -x^2 \in \mathbb{R}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}, \quad x \in \mathbb{R}$$

Hence the two solutions are the same.

E.g: Find a power series solution to the I.V.P. You only need to find the first four nonzero terms of the power series.

$$(1+x^2)y''-y'+y=0,\quad y(0)=1,\quad y'(0)=-1.$$

A: Step 1: Write y as a power series, and compute y', y''.

By Taylor's Thm,

$$y=\sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow a_n = \frac{y^{(n)}(0)}{n!}$$

$$\Rightarrow a_0 = \frac{y(0)}{0!} = y(0) = 1$$

$$a_1 = \frac{y'(0)}{1!} = -1$$

Lecture 21

write $y=\sum_{n=0}^{\infty} a_n x^n$.

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow y'' = (y')'$$

$$=\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)'$$

$$=\sum_{n=1}^{\infty}(n a_n x^{n-1})'$$

$$=\sum_{n=1}^{\infty} n(n-1)a_n x^{n-2}$$

$$=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

When $n=1$, $n(n-1)a_n x^{n-2}=0$

Step 2. Plug in y, y', y'' to the D.E

$$(1+x^2)y''-y'+y=0$$

$$(1+x^2)\underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{y''} - \underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{y'} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{y} = 0$$

$\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n$$

Next we shift all summation indices to make general term be x^k

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \xrightarrow[\substack{\text{let } k=n-2 \\ n=k+2}]{} \sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2}x^k = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n \xrightarrow[\text{let } k=n]{} \sum_{k=2}^{\infty} k(k-1)a_k x^k$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \xrightarrow[\substack{\text{let } k=n-1 \\ \Rightarrow n=k+1}]{} \sum_{k=0}^{\infty} (k+1)a_{k+1} x^k$$