Lecture 23

Plan: • Finish §8.3

Solve D.E using power series

• Start chapter 9 by recalling some basic linear algebra

E.g: Find a power series solution to the I.V.P. You only need to find the first four nonzero terms of the power series.

$$(1+x^2)y'' - y' + y = 0, \quad y(0)=1, \quad y'(0) = -1.$$

A: Step 1: Write y as a power series, and compute y', y''.

By Taylor's Thm, $x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow a_n = \frac{y^{(n)}(0)}{n!}$$

$$\Rightarrow a_0 = \frac{y(0)}{0!} = y(0) = 1$$

$$a_1 = \frac{y'(0)}{1!} = -1$$

Lecture 21

write $y = \sum_{n=0}^{\infty} a_n x^n$.

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow y'' = (y')'$$

$$= \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)'$$

$$= \sum_{n=1}^{\infty} (n a_n x^{n-1})'$$

$$= \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

When $n=1$, $n(n-1)a_n x^{n-2} = 0$

Step 2. Plug in y, y', y'' to the D.E

$$(1+x^2)y'' - y' + y = 0$$

$$(1+x^2)\underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{y''} - \underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{y'} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{y} = 0$$

$\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n$$

Next we shift all summation indices to make general term be x^k

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \xrightarrow[\substack{\text{Let } k=n-2\\ n=k+2}]{} \sum_{k+2=2}^{\infty}(k+2)(k+1)a_{k+2}x^k = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n \xrightarrow[\text{Let } k=n]{} \sum_{k=2}^{\infty} k(k-1)a_k x^k$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \xrightarrow[\substack{\text{Let } k=n-1\\ \Rightarrow n=k+1}]{} \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k$$

$$\sum_{n=0}^{\infty} a_n x^n \xrightarrow{\text{Let } n=k} \sum_{k=0}^{\infty} a_k x^k.$$

$$\Longrightarrow$$

$$\underbrace{\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k}_{①} + \underbrace{\sum_{k=2}^{\infty} k(k-1)a_k x^k}_{②} - \underbrace{\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k}_{③} + \underbrace{\sum_{k=0}^{\infty} a_k x^k}_{④} = 0$$

Step 3. Compute the coefficients of each x^k-term and they must be equal to 0.

Hint:

$k=0 \Leftarrow x^0$-term: $\overset{①}{2\cdot 1\cdot a_2} \overset{③}{- a_1} + \overset{④}{a_0} = 0$ (1)

② : No contribution

$k=1 \Leftarrow x^1$-term: $\overset{①}{3\cdot 2\cdot a_3} \overset{③}{- 2a_2} + \overset{④}{a_1} = 0$ (2)

②: No contribution

x^k-term for $k \geq 2$:

$$\underbrace{(k+2)(k+1)a_{k+2}}_{①} + \underbrace{k(k-1)a_k}_{②} - \underbrace{(k+1)a_{k+1}}_{③} + \underbrace{a_k}_{④} = 0 \quad (3)$$

Recall $\begin{cases} 2a_2 - a_1 + a_0 = 0 & (1) \\ \text{initial conditions: } a_0 = 1, \quad a_1 = -1 \end{cases}$

Goal: first 4 nonzero terms: a_0, a_1, a_2, a_3

$\Rightarrow \quad 2a_2 = a_1 - a_0$

$= -1 - 1 = -2$

$\Rightarrow \quad a_2 = -1.$

Recall $3 \cdot 2 \cdot a_3 - 2a_2 + a_1 = 0 \qquad (2)$

$\Rightarrow 6a_3 = 2a_2 - a_1 = -2 + 1 = -1$

$\Rightarrow \quad a_3 = -\frac{1}{6}$

Hence $y = \sum_{n=0}^{\infty} a_n x^n$

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \sum_{n=4}^{\infty} a_n x^n$

$= \boxed{1 - x - x^2 - \frac{1}{6}x^3 + \text{H.O.T}}$

higher order terms

E.g Solve the following I.V.P problem using power series. You need to find the first three nonzero terms in the power series:

$$\begin{cases} 2y'' + xy' + y = 0 \\ y(0) = 0, \; y'(0) = 6 \end{cases}$$

A: Step 1: write y as a power series (Identify a_0, a_1), and compute y', y''.

write $y = \sum_{n=0}^{\infty} a_n x^n$

By Taylor's thm, $\begin{cases} y(0) = 0 \Rightarrow a_0 = 0 \\ y'(0) = 6 \Rightarrow \frac{a_1}{1!} = 6 \end{cases}$

$\Rightarrow a_1 = 6$

And $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Step 2. plug y, y', y'' into the D.E

$$2y'' + xy' + y = 0$$

$$\Rightarrow$$

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Shift all summation indices to make the general term be "x^k".

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} \xrightarrow[\substack{\text{let } k=n-2 \\ \Rightarrow n=k+2}]{} \sum_{k=0}^{\infty} 2(k+2)(k+1) a_{k+2} x^k$$

$$\sum_{n=1}^{\infty} n a_n x^n \xrightarrow[\text{let } n=k]{} \sum_{k=1}^{\infty} k a_k x^k$$

$$\sum_{n=0}^{\infty} a_n x^n \xrightarrow[\text{let } n=k]{} \sum_{k=0}^{\infty} a_k x^k$$

$$\Rightarrow$$

$$\underbrace{\sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}x^k}_{①} + \underbrace{\sum_{k=1}^{\infty} k a_k x^k}_{②} + \underbrace{\sum_{k=0}^{\infty} a_k x^k}_{③} = 0$$

Step 3: Compute the coefficients of x^k on LHS and they must be equal 0.

$k=0$

$$x^0: \underset{①}{2\cdot 2\cdot 1\cdot a_2} + \underset{③}{a_0} = 0$$

② : No contribution

$$\Rightarrow 4a_2 + a_0 = 0$$

But $a_0 = 0 \Rightarrow a_2 = 0$

x^k, $k \geq 1$:

$$\underbrace{2(k+2)(k+1)a_{k+2}}_{①} + \underbrace{k a_k}_{②} + \underbrace{a_k}_{③} = 0$$

$$\Rightarrow \quad 2(k+2)(k+1)a_{k+2} = -(k+1)a_k, \; k \ge 1$$

Let $k=1$, $\Rightarrow$ $2 \cdot 3 \cdot 2 a_3 = -2a_1$

$$\Rightarrow \quad a_3 = -\frac{1}{6}a_1$$

Recall $a_1 = 6 \Rightarrow a_3 = -1$

$a_0 = 0$
$a_1 = 6$ ✓
$k=0 \rightarrow a_2 = 0$
$k=1 \rightarrow a_3 = -1$ ✓
$k=2 \rightarrow a_4 = 0$
$k=3 \rightarrow a_5 = \frac{1}{10}$ ✓

Let $k=2 \Rightarrow$

$$2 \cdot 4 \cdot 3\, a_4 = -3a_2$$

$$a_2 = 0 \Rightarrow a_4 = 0$$

Let $k=3$. $\Rightarrow$

$$2 \cdot 5 \cdot 4\, a_5 = -4a_3$$

$$\Rightarrow \quad a_5 = -\frac{1}{10}a_3 = \frac{1}{10}$$

Hence $y = \sum_{n=0}^{\infty} a_n x^n$

$$= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \sum_{n=6}^{\infty} a_n x^n$$

$$= \boxed{6x - x^3 + \frac{1}{10}x^5 + H.O.T}$$

Chapter 9

In this chapter, we will use linear algebra to help solve D.E. For that, we recall some basic concepts from linear algebra.

- A matrix is a rectangular array of numbers, arranged in rows and columns.

an $m\times n$ matrix is a matrix that has m rows and n columns:

a_{ij} is called an entry of A.

$$A := \begin{bmatrix} a_{11}, & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Sometimes we write A as

$$A = [a_{ij}]_{1\le i\le m,\ 1\le j\le n} \quad \text{or simply } A = [a_{ij}].$$

- A is called a square matrix if $m=n$ (that is, A has the same # of rows and columns) diagonal v.s off diagonal

- A is called a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.

E.g $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

are all diagonal

- A is called a column vector if $n=1$. (that is, A has only one column).

E.g: $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

A is called a row vector if $m=1$. (that is, A has only one row)

E.g: $A = [1, 2, 3, 4]$.

- A is called a zero matrix if all its entries are zero.

E.g $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$: zero 2x2 matrix

- an $n \times n$ matrix is called an identity matrix

  if $\begin{cases} a_{ii} = 1 & \text{for } 1 \le i \le n \\ \text{on diagonal, entry} = 1 & \\ a_{ij} = 0 & \text{for } 1 \le i \ne j \le n \\ \text{off diagonal, entry} = 0 & \end{cases}$

E.g $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: 2x2 identity matrix

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: 3x3 identity matrix

Ex: Read the next two pages by yourself.

we will continue from here next time.

[redacted]

Linear systems or equations can be represented using matrices and vectors. And it turns out that such representation is helpful even in solving the equation. Therefore, let us discuss matrices and vectors in the next subsection.

6.1 Matrices and vectors

A **matrix** is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. An $m \times n$ matrix is with m rows and n columns:

$$A := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

We can simply write $[a_{ij}] \in \mathbb{R}^{m\times n}$ to denote the matrix.

Square matrices: $m = n$.

Diagonal matrices: $a_{ij} = 0$ for all $i \neq j$.

(Column) Vectors: $m \times 1$ matrices. **(Row) Vectors**: $1 \times m$ matrices.

Zero matrix: $a_{ij} = 0$ for all i, j, denoted as **0**.

Can you write equation in Example 6.0.1 into a matrix form where the matrix only depend on the free variable?

6.1.1 Algebra of Matrices

Scalar Multiplication. Let $r \in \mathbb{R}$ and $A = [a_{ij}]$ be a matrix. Then

$$rA = [ra_{ij}].$$

We write

$$-A := (-1)A = [-a_{ij}]$$

Matrix Addition. We can add up two $m \times n$ matrices. Suppose $A = [a_{ij}]$, $B = [b_{ij}]$ are two $m \times n$ matrices, then

$$A + B = [a_{ij} + b_{ij}], \quad A - B = [a_{ij} - b_{ij}].$$

If two matrices have different numbers of rows or columns, we can not add the matrices up.

Matrix Multiplication. Let $A = [a_{ij}]$ be a $m \times n$ matrix and let

$$B = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}$$

be a n-dimensional column vector. Then

$$AB = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + ...a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + ...a_{2n}b_n \\ ... \\ a_{m1}b_1 + a_{m2}b_2 + ...a_{mn}b_n \end{bmatrix}.$$

For example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+2+6 \\ 0+5+12 \\ 0+8+18 \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \\ 26 \end{bmatrix}.$$

In general, we are able to define AB if A is an $m \times n$ matrix and B is an $n \times p$ matrix:

$$AB = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & ... & b_{1p} \\ b_{21} & b_{22} & ... & b_{2p} \\ b_{n1} & b_{n2} & ... & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + ...a_{1n}b_{n1} & ... & a_{11}b_{1p} + a_{12}b_{2p} + ...a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}b_{21} + ...a_{2n}b_{n1} & ... & a_{21}b_{1p} + a_{22}b_{2p} + ...a_{2n}b_{np} \\ ... & & \\ a_{m1}b_{11} + a_{m2}b_{21} + ...a_{mn}b_{n1} & ... & a_{m1}b_{1p} + a_{m2}b_{2p} + ...a_{mn}b_{np} \end{bmatrix}.$$

If we denote $C := AB$ then $C = [c_{ij}]$ is a $m \times p$ matrix and for $i = 1, ..., m, j = 1, ..., p$

$$c_{ij} = \Sigma_{k=1}^{n} a_{ik}b_{kj}.$$

Theorem 6.1. *Suppose A, B, C are matrices and r is a number. The following holds as long as they are well-defined:*

$$A + B = B + A, \quad r(A + B) = rA + rB,$$

$$(AB)C = A(BC), \quad A(B + C) = AB + AC.$$

Example 6.1.1. *Let A, B be two $n \times n$ matrices. Remove the bracket of $(A + B)^2$.*

Solution.

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

Note that this is not the same as $A^2 + 2AB + B^2$. In matrices multiplication, $AB \neq BA$. □