Lecture 27

plan: §9.7 nonhomogeous linear system.

In this lecture, we try to solve nonhomogeneous linear system of the form:

$$\vec{x}' = \underset{n\times n}{A}\vec{x} + \underset{n\times 1}{\vec{f}(t)}$$

$\vec{x}: n\times 1$

We will discuss two methods to solve it.

(I) Undetermined coefficients

(II) variation of parameters

(I) Undetermined coefficients

E.g. Find the general solution to

$$\vec{x}' = A\vec{x} + t\vec{g}, \text{ where}$$

$\vec{f} = t\vec{g}$

$$A = \underset{3\times 3}{\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}}, \quad \vec{g} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

$$= \begin{bmatrix} t \\ -2t \\ 2t \end{bmatrix}$$

P_2

A: Step 1: Find the general solution $\vec{x}_h$ to the associated homogeneous linear system:

$$\vec{x}' = A\vec{x}$$

We did this already in Lecture 26.

$$\vec{x}_h = c_1 e^{3t}\begin{bmatrix}0\\1\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\0\\1\end{bmatrix} + c_3 e^{-3t}\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Step 2: Find a particular soln $\vec{x}_p$ to the nonhomogeneous linear system.

Hint: Try the test vector function

$$\vec{x}_p = t\vec{a} + \vec{b} = t\underbrace{\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}}_{\vec{a}} + \underbrace{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}}_{\vec{b}}$$

plug in to " $\vec{x}' = A\vec{x} + \vec{f}$ ".

LHS: Note $\vec{x}_p' = \vec{a}$

(why? Indeed, $\vec{x}_p = t\vec{a} + \vec{b} = \begin{bmatrix}ta_1+b_1\\ta_2+b_2\\ta_3+b_3\end{bmatrix}$

$$\Rightarrow \vec{x}_p' = \begin{bmatrix}(ta_1+b_1)'\\(ta_2+b_2)'\\(ta_3+b_3)'\end{bmatrix} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$$)

RHS:

$$A\vec{x}_p + \vec{f} = A\vec{x}_p + t\vec{g}$$
$$= A(t\vec{a} + \vec{b}) + t\vec{g}$$
$$= t(A\vec{a} + \vec{g}) + A\vec{b}$$

LHS = RHS $\Rightarrow$ $\vec{a} = t(A\vec{a} + \vec{g}) + A\vec{b}$

Compare: t-term:

$$\vec{0} = A\vec{a} + \vec{g} \Rightarrow A\vec{a} = -\vec{g} \quad (1)$$

Constant-term: $\vec{a} = A\vec{b}$. (2)

$$(1) \Rightarrow \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a_1 - 2a_2 + 2a_3 = -1 & (1) \\ -2a_1 + a_2 + 2a_3 = 2 & (2) \\ 2a_1 + 2a_2 + a_3 = -2 & (3) \end{cases}$$

$$(1)+(3) \Rightarrow 3a_1 + 3a_3 = -3$$
$$(1)+2(2) \Rightarrow -3a_1 + 6a_3 = 3$$
$$\Rightarrow \begin{cases} a_1 = -1 \\ a_3 = 0 \end{cases}$$

$\Rightarrow$ $a_2 = 0$

Hence $\vec{a} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

② $A\vec{b} = \vec{a} \Rightarrow$

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} b_1 - 2b_2 + 2b_3 = -1 \\ -2b_1 + b_2 + 2b_3 = 0 \\ 2b_1 + 2b_2 + b_3 = 0 \end{cases} \Rightarrow \underset{\uparrow \text{ E.X}}{\cdots} \Rightarrow \begin{cases} b_1 = -\frac{1}{9} \\ b_2 = \frac{2}{9} \\ b_3 = -\frac{2}{9} \end{cases}$$

$$\Rightarrow \vec{b} = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix}$$

Hence $\vec{x}_p = t\vec{a} + \vec{b} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix}$

Step 3: Write down the general soln to the nonhomogeneous linear system:

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

$\vec{x}_h$: the general soln to the assoc. homogeneous system

$$\Rightarrow$$

$$\vec{x} = c_1 e^{3t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix}$$

$$\vec{x}_p = t\vec{a} + \vec{b}$$

Given $\vec{x}' = A\vec{x} + \vec{f}(t)$. $(*)$

It is required to know how to solve $(*)$ for two types of $\vec{f}(t)$:

① $\vec{f}(t) = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$, $A_1, \cdots, A_n \in \mathbb{R}$

Algorithm:

Attempt 1. Try the test function

$\vec{x}_p = \vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $a_1, \cdots, a_n \in \mathbb{R}$

will be enough to use Attempt 1 in HW and Exam

Attempt 2. If the above does not work,

try the new test function

$$\vec{x}_p = t\vec{a} + \vec{b} = t\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Attempt 3. If it still does not work,

try

$$\vec{x}_p = t^2\vec{a} + t\vec{b} + \vec{c} = t^2\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + t\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Attempt 4. ....

Attempt 5. ....

⋮

② $\vec{f}(t) = t\begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}$, $A_1, \cdots, A_n, B_1, \cdots, B_n \in \mathbb{R}$

$B_1, \cdots, B_n$ can be 0

Algorithm:

Attempt 1: Try the test function

$$\vec{x}_p = t\vec{a} + \vec{b} = t\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

will be enough to use attemp 1 in HW and Exam

Attempt 2: If the above does not work, try the new test function

$$\vec{x}_p = t^2\vec{a} + t\vec{b} + \vec{c} = t^2\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + t\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Attempt 3: If still does not work, try

$$\vec{x}_p = t^3\vec{a} + t^2\vec{b} + t\vec{c} + \vec{d} = t^3\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \cdots + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

Attempt 4: - - …

⋮

Recall in the E.g we did,

there $\vec{f} = t\begin{bmatrix}1\\-2\\2\end{bmatrix} = t\begin{bmatrix}1\\-2\\2\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix}$

Hence our 1st attempt is:

to use the test function:

$$\vec{x}_p = t\vec{a} + \vec{b} = t\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} + \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}.$$

And it worked.

(II) Variation of Parameters

Idea: To solve a nonhomogeneous linear

system $\vec{x}' = A\vec{x} + \vec{f}(t)$ (1)

NOT important

A: $n \times n$ matrix

we first consider its associated homogeneous linear system:

$$\vec{x}' = A\vec{x} \qquad (2)$$

Assume a fundamental solution set to (2) is given by

$$\{\vec{x}_1, \cdots, \vec{x}_n\}$$

write $\vec{x}_1 = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}_{n\times 1}$, $\cdots$, $\vec{x}_n = \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}_{n\times 1}$

Set

$$X = [\vec{x}_1, \cdots, \vec{x}_n] = \underbrace{\begin{bmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{bmatrix}}_{n\times n}$$

<u>claim</u>: $\underset{n\times n}{X'} = \underset{n\times n}{A}\,\underset{n\times n}{X}$

Pf of claim: E. X

Hint: use the fact that

$$\underbrace{\vec{x}_1' = A\vec{x}_1, \quad \cdots, \quad \vec{x}_n' = A\vec{x}_n}$$

because $\vec{x}_1, \cdots, \vec{x}_n$ are solns to " $\vec{x}' = A\vec{x}$ "

Key idea: We seek a particular solution to the nonhomogeneous system " $\vec{x}' = A\vec{x} + \vec{f}$ ".

of the form: $\vec{x}_p = \underset{n\times n}{X}\ \underset{n\times 1}{\vec{v}(t)}$,

$X = [\vec{x}_1, \cdots, \vec{x}_n]$

where $\vec{v}(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix}$.

Suppose $\vec{x}_p = X\vec{v}(t)$ is a soln to

" $\vec{x}' = A\vec{x} + \vec{f}$ " $\Longrightarrow$ " $\vec{x}_p' = A\vec{x}_p + \vec{f}$ "

Compute:

$$\vec{x}_p' = (X\vec{v})' = X'\vec{v} + X\vec{v}'$$

(Here we used the formula:

$$\frac{d}{dt}\underbrace{(\underset{m\times n}{A(t)} \cdot \underset{n\times p}{B(t)})}_{m\times p} = \left(\frac{d}{dt}A(t)\right)\cdot B(t) + A(t)\left(\frac{d}{dt}B(t)\right)$$

)

$$A\vec{x}_p + \vec{f} = A(X\vec{v}) + \vec{f}$$
$$= AX\vec{v} + \vec{f}$$

$$\Rightarrow \quad X'\vec{v} + X\vec{v}' = AX\vec{v} + \vec{f}$$

But by the claim: $X' = AX$

$$\Rightarrow \quad X'\vec{v} = AX\vec{v}$$

Hence $X\vec{v}' = f$

$$\Rightarrow \quad \vec{v}' = X^{-1} f$$

(Recall $X = \underbrace{[\vec{x}_1, \cdots, \vec{x}_n]}_{\text{are linearly independent}}{}_{n\times n}$

$\Rightarrow \text{rank}(X) = n \Rightarrow X$ is invertible)

Integrating $\Rightarrow$

$$\vec{v} = \int X^{-1} f \, dt$$

$$\Rightarrow \quad \vec{x}_p = X\vec{v} = X \int X^{-1} f \, dt$$

This motivates the following algorithm of variation of parameters.

☆ Algorithm:

Find a particular soln to "$\vec{x}' = A\vec{x} + \vec{f}$".

Step 1: Consider the associated homogeneous system $\vec{x}' = A\vec{x}$.

Find a fundamental soln set to "$\vec{x}' = A\vec{x}$":

$$\vec{x}_1 = e^{\lambda_1 t}\vec{v}_1, \quad \cdots, \quad \vec{x}_n = e^{\lambda_n t}\vec{v}_n,$$

Step 2: Form an $n \times n$ matrix

$$X = [\vec{x}_1, \cdots, \vec{x}_n]$$

and compute X^{-1}.

Step 3. Compute $\vec{v}(t) = \int X^{-1}\vec{f}\,dt$

and $\vec{x}_p = X(t)\vec{v}(t)$ is a particular soln to "$\vec{x}' = A\vec{x} + \vec{f}$".

Step 4: If we need to find the general soln to "$\vec{x}' = A\vec{x} + \vec{f}$".

then it is

$$\vec{x} = \vec{x}_h + \vec{x}_p = \underbrace{c_1\vec{x}_1 + \cdots + c_n\vec{x}_n}_{\vec{x}_h} + \vec{x}_p$$

E.g. Find the general soln to

$$\vec{x}' = \underbrace{\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}}_{A}{}_{2\times 2} \vec{x} + \underbrace{\begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}}_{\vec{f}}$$

A: Step 1: Find a fundamental soln set to "$\vec{x}' = \underbrace{\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}}_{A} \vec{x}$"

We indeed already did this in Lecture 26.

We briefly repeat it here:

Compute

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 3 \\ -1 & \lambda+2 \end{vmatrix} = \lambda^2 - 1$$

$\Rightarrow$ eigenvalues of A: $\lambda_1 = -1$, $\lambda_2 = 1$

Then we compute the corresponding eigenvector $\vec{v}_1$, $\vec{v}_2$, respectively.

① $(\lambda_1 I - A)\vec{v} = 0$

$$\Rightarrow \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow y = x$

pick $x = 1$. $\Rightarrow y = 1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

② $(\lambda_2 I - A)\vec{v} = 0$

$$\Rightarrow \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow x = 3y$

pick $y = 1$. $\Rightarrow x = 3 \Rightarrow \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Summarize:

$\lambda_1 = -1 \longrightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \longrightarrow \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\Rightarrow$ a fundamental soln set:

$$\vec{x}_1 = e^{\lambda_1 t}\vec{v}_1 = e^{-t}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}e^{-t}\\e^{-t}\end{bmatrix}$$

$$\vec{x}_2 = e^{\lambda_2 t}\vec{v}_2 = e^{t}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}3e^{t}\\e^{t}\end{bmatrix}$$

Step 2: Form a 2×2 matrix

$$X = [\vec{x}_1 \ \ \vec{x}_2]$$

$$= \begin{bmatrix}e^{-t} & 3e^{t}\\ e^{-t} & e^{t}\end{bmatrix}$$

Then compute X^{-1}.

Very useful formula:

If $X = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$,

then $X^{-1} = \frac{1}{\det(X)}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

given $\det(X) = ad-bc \neq 0$

Recall
$$X = \begin{bmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{bmatrix}$$

$$\Rightarrow \quad \det(X) = e^{-t}e^{t} - e^{-t}\,3e^{t}$$
$$= 1 - 3 = -2$$

By the formula,
$$X^{-1} = \frac{1}{-2}\begin{bmatrix} e^{t} & -3e^{t} \\ -e^{-t} & e^{-t} \end{bmatrix}$$

Step 3: compute
$$\vec{v} = \int X^{-1}\vec{f}\,dt$$

Recall $\vec{f} = \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$

Note
$$X^{-1}\vec{f} = -\frac{1}{2}\begin{bmatrix} e^{t} & -3e^{t} \\ -e^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$$
$$= -\frac{1}{2}\begin{bmatrix} e^{3t} - 3e^{t} \\ -e^{t} + e^{-t} \end{bmatrix}$$

$\Rightarrow$

$$\vec{v} = \int -\frac{1}{2}\begin{bmatrix} e^{3t} - 3e^{t} \\ -e^{t} + e^{-t} \end{bmatrix} dt$$

$$= -\frac{1}{2}\int \begin{bmatrix} e^{3t}-3e^{t} \\ -e^{t}+e^{-t} \end{bmatrix} dt$$

$$= -\frac{1}{2} \begin{bmatrix} \int (e^{3t}-3e^{t})dt \\ \int (-e^{t}+e^{-t})dt \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} \frac{1}{3}e^{3t}-3e^{t}+C_1 \\ -e^{t}-e^{-t}+C_2 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} \frac{1}{3}e^{3t}-3e^{t} \\ -e^{t}-e^{-t} \end{bmatrix}$$

Choose $C_1 = 0$

$C_2 = 0$

$\vec{v}$

$$\Rightarrow \vec{x}_p = X\vec{v}$$

$$= \begin{bmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{bmatrix} \left(-\frac{1}{2} \begin{bmatrix} \frac{1}{3}e^{3t}-3e^{t} \\ -e^{t}-e^{-t} \end{bmatrix} \right)$$

$$= -\frac{1}{2} \begin{bmatrix} \left(\frac{1}{3}e^{2t}-3\right)-3e^{2t}-3 \\ \frac{1}{3}e^{2t}-3-e^{2t}-1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -\frac{8}{3}e^{2t} - 6 \\ -\frac{2}{3}e^{2t} - 4 \end{bmatrix}$$

Step 4. Write the general soln to the given nonhomogeneous system

$$\vec{x} = \overbrace{c_1\vec{x}_1 + c_2\vec{x}_2}^{\vec{x}_h} + \vec{x}_p$$

$$= c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \vec{x}_p$$

$$= c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -\frac{8}{3}e^{t} - 6 \\ -\frac{2}{3}e^{2t} - 4 \end{bmatrix}$$

Note: An additional review class (Lecture 28) will be given.