Lecture 28

plan: Review class for the final

We will discuss the practice problems

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1. Solve $y(t)$ from the following integral equation:

$$y(t) = 1 + t - \int_0^t (t-x)\, y(x)\, dx$$

Recall: §7.8

<u>Defn</u>: Convolution of f, g

$$(f * g)(t) = \int_0^t f(t-x)\, g(x)\, dx$$

properties: • $f * g = g * f$

☆ • If $F(s) = \mathcal{L}\{f\}(s)$, $G(s) = \mathcal{L}\{g\}(s)$

then $\mathcal{L}\{f * g\}(s) = F(s)\, G(s)$.

equivalently, $\mathcal{L}^{-1}\{F(s)\,G(s)\}(t) = (f*g)(t)$.

A: Step 1: Rewrite "$y(t) = 1+t-\int_0^t (t-x)\,y(x)\,dx$"

by observing the convolution. Then apply Laplace transform $\mathcal{L}$ to the eqn

Note $y(t) = 1+t-\int_0^t (t-x)\,y(x)\,dx$

$\Rightarrow \quad y(t) = 1+t - t*y(t)$

Apply $\mathcal{L}$ to the above eqn $\Rightarrow$

$$\mathcal{L}\{y\}(s) = \mathcal{L}\{1\} + \mathcal{L}\{t\} - \mathcal{L}\{t*y(t)\}$$

Recall table from §7.2

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Rightarrow \quad \mathcal{L}\{t\} = \frac{1}{s^2}$$

write $Y = \mathcal{L}\{y\}$. $\Rightarrow$

$$Y = \frac{1}{s} + \frac{1}{s^2} - \mathcal{L}\{t\}\mathcal{L}\{y(t)\}$$

$$\Rightarrow Y = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^2}Y$$

$$\Rightarrow (s^2+1)Y = s+1$$

$$\Rightarrow Y = \frac{s+1}{s^2+1}$$

Step 2: Compute $y(t) = \mathcal{L}^{-1}\{Y\}$

Since $Y = \mathcal{L}\{y\}$

$$\Rightarrow y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1} + \frac{1}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \cos t + \sin t$$

Here we used Table in §7.2

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \quad \text{for } b \in \mathbb{R}$$

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+1}$$

Remark:

- You may like to review Q4 in HW 6. which is similar to the above problem

- Reminder: To solve problems from 7.8 7.9, you might need knowledge from ealier sections (In particular, 7.2, 7.3).

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2. Find the solution of the I.V.P

Q1 in HW7

$$y'' + 2y' + 2y = \delta(t-\pi),\ y(0)=1,\ y'(0)=0$$

Recall §7.9

- "δ" is a function $\mathbb{R} \to \mathbb{R} \cup \{\infty\}$ satisfying

① $\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$

② $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$

if f is continuous in some interval containing 0.

- $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)$

- for $a \geq 0$, $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$.

Recall §7.3

- $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$.
- $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$

A: Step 1: Apply $\mathcal{L}$ to the eqn

" $y'' + 2y' + 2y = \delta(t-\pi)$ "

$\Rightarrow$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\}$$

Write $Y = \mathcal{L}\{y\}$. Note by §7.3

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$
$$= s^2Y - s;$$

Recall initial condition
$y(0) = 1$
$y'(0) = 0$

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$
$$= sY - 1$$

By §7.9. $\mathcal{L}\{\delta(t-\pi)\} = e^{-\pi s}$

$$\Rightarrow \quad s^2Y - s + 2(sY-1) + 2Y = e^{-\pi s}$$

$$\Rightarrow \quad (s^2+2s+2)Y = s+2+e^{-\pi s}$$

$$\Rightarrow \quad Y = \frac{s+2+e^{-\pi s}}{s^2+2s+2}$$

Step2: Compute $y = \mathcal{L}^{-1}\{Y\}$

Note $y = \mathcal{L}^{-1}\{Y\}$

$$= \mathcal{L}^{-1}\left\{\frac{s+2+e^{-\pi s}}{s^2+2s+2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}\right\}$$

Note
(s^2+2s+1)
$= (s+1)^2+1$

Recall §7.2

$$\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$$

Let $a=-1$, $b=1$

$$\Rightarrow \mathcal{L}\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1}$$

Recall §7.6

u: step function

If $c>0$, given $F = \mathcal{L}\{f\}$

$$\Rightarrow \mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}F(s)$$

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)u(t-c)$$

Now $\underbrace{\frac{1}{(s+1)^2+1}}_{F(s)} = \mathcal{L}\{\underbrace{e^{-t}\sin t}_{f(t)}\}$.

Let $c=\pi$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-\pi s}\frac{1}{(s+1)^2+1}\} = f(t-\pi)\,u(t-\pi)$$

$$= e^{-(t-\pi)}\sin(t-\pi)\,u(t-\pi)$$

Since $f(t) = e^{-t}\sin t$

Replacing all "t" by "$t-\pi$".

$$\Rightarrow f(t-\pi) = e^{-(t-\pi)}\sin(t-\pi)$$

$$= e^{-(t-\pi)}(-\sin t)$$

Hence

$$y = e^{-t}\sin t + e^{-t}\cos t + e^{-(t-\pi)}\sin(t-\pi)u(t-\pi)$$

3. Solve the I.V.P using power series centered at 0. Find the first four nonzero terms of the power series. Then find $y^{(3)}(0)$.

$$y'' - xy' - y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

§8.3

A: Step 1: write y as a power series, (and identify a_0, a_1), and compute y', y''.

write $y = \sum_{n=0}^{\infty} a_n x^n$

By Taylor's thm, $a_n = \dfrac{y^{(n)}(0)}{n!}$

$$\Rightarrow \quad a_0 = y(0) = 2$$
$$a_1 = y'(0) = -1$$

Then we compute $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

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$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Step 2: plug into the D.E

$$y'' - xy' - y = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{1} - \underbrace{\sum_{n=1}^{\infty} n a_n x^{n}}_{2} - \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{3} = 0$$

Next we shift all summation indices to make the general term be x^k

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \xrightarrow[\iff n=k+2]{\text{Let } k=n-2} \sum_{k+2=2}^{\infty} (k+2)(k+1)a_{k+2}x^k = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k$$

$$\sum_{n=1}^{\infty} n a_n x^n \xrightarrow[\iff n=k]{\text{Let } k=n} \sum_{k=1}^{\infty} k a_k x^k$$

$$\sum_{n=0}^{\infty} a_n x^n \xrightarrow[\iff n=k]{\text{Let } k=n} \sum_{k=0}^{\infty} a_k x^k$$

$\Longrightarrow$

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty}k a_k x^k - \sum_{k=0}^{\infty}a_k x^k = 0$$

① ② ③

Step 3. Compute the coefficients of each x^k-term and they must equal to 0.

$k=0$

x^0-term: $\underset{①}{2\cdot 1\cdot a_2} - \underset{③}{a_0} = 0$

$$\Longrightarrow a_2 = \frac{1}{2}a_0 = \frac{1}{2}\cdot 2 = 1.$$

for $k\geq 1$

x^k-term: $(k+2)(k+1)a_{k+2} - k a_k - a_k = 0$

$$\Longrightarrow (k+2)(k+1)a_{k+2} = (k+1)a_k$$

Let $k=1 \Rightarrow 3\cdot 2\cdot a_3 = 2a_1$

$$\Longrightarrow a_3 = \frac{1}{3}a_1 = -\frac{1}{3}$$

Recall we need to find the first four nonzero terms

$$\left| \begin{array}{l} a_0 = 2 \\ a_1 = -1 \\ a_2 = 1 \\ a_3 = -\frac{1}{3} \end{array} \right|$$

Next we compute $y^{(3)}(0)$:

By Taylor's thm, $a_n = \dfrac{y^{(n)}(0)}{n!}$

Let $n=3$, $\Rightarrow a_3 = \dfrac{y^{(3)}(0)}{3!}$

$$\Rightarrow y^{(3)}(0) = 3!\, a_3$$

$$= (3!) - \frac{1}{3} = -2$$

4: Find the general solution of the homogeneous linear system:

$$\vec{x}' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \vec{x}$$

$A =$ Write $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

Step 1: Find the eigenvalues of A by solving $\det(\lambda I_3 - A) = 0$

Note

$$\lambda I_3 - A = \begin{bmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -4 & 0 & \lambda-1 \end{bmatrix}$$

$\Rightarrow$

$$\det(\lambda I_3 - A) = (\lambda-1)\begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix} + (-1)\begin{vmatrix} 0 & \lambda-1 \\ -4 & 0 \end{vmatrix}$$

$$= (\lambda-1)(\lambda-1)^2 - 1 \cdot 4 \cdot (\lambda-1)$$

$$= (\lambda-1)^3 - 4(\lambda-1)$$

$$= (\lambda-1)((\lambda-1)^2 - 4)$$

$$= (\lambda-1)(\lambda+1)(\lambda-3)$$

Let $\det(\lambda I_3 - A) = 0 \Rightarrow$

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 3$$

They are the eigenvalues of A.

Step 2: Find the eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ corresponding to $\lambda_1, \lambda_2, \lambda_3$.

① Find an eigenvector $\vec{v}_1$ corresponding to $\lambda_1 = 1$. For that, we solve $(\lambda_1 I_3 - A)\vec{v}_1 = \vec{0}$. Write $\vec{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -z = 0 \Rightarrow z = 0 \\ -4x = 0 \Rightarrow x = 0 \end{cases}$$

y is free to choose

we pick $y=1 \Rightarrow$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

② Find $\vec{v}_2$ corresponding to $\lambda_2 = -1$

Solve $(\lambda_2 I_3 - A)\vec{v}_2 = 0$

write $\vec{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 0 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -2x - z = 0 & ① \\ -2y = 0 & ② \\ -4x - 2z = 0 & ③ \end{cases}$$

Note

① $\Rightarrow$ ③

$$\Rightarrow \begin{cases} 2x + z = 0 \\ 2y = 0 \end{cases} \Rightarrow \begin{cases} z = -2x \\ y = 0 \end{cases}$$

choose $x = 1$, $\Rightarrow z = -2$

Hence $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

③ Find $\vec{v}_3$ corresponding to $\lambda_3 = 3$

We solve

$$(\lambda_3 I_3 - A)\vec{v}_3 = 0$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} 2x - z = 0 & ① \\ 2y = 0 & ② \\ -4x + 2z = 0 & ③ \end{cases}$$

Note

① $\Rightarrow$ ③

$$\Rightarrow \begin{cases} 2x - z = 0 \\ 2y = 0 \end{cases} \Rightarrow \begin{cases} z = 2x \\ y = 0 \end{cases}$$

Choose $x = 1 \Rightarrow z = 2 \Rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Summarize:

$$\lambda_1 = 1 \longrightarrow \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1 \longrightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\lambda_3 = 3 \rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Step 3: write down the general soln

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + c_3 e^{\lambda_3 t} \vec{v}_3$$

$$= c_1 e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

5. Use the method of undetermined coefficients to find the general soln:

$$\vec{x}' = \underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}}_{A} \vec{x} + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{f}}$$

A: Step 1: Find the general soln to the associated homogeneous system:

$$\vec{x}' = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \vec{x}$$

$\uparrow A$

Find eigenvalues of A.

$$\det(\lambda I_2 - A) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 3 & -2 \\ 1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda-3) + 2 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0$$

$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 2 \leftarrow$ eigenvalues of A.

Find corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$.

Solve $(\lambda_1 I_2 - A)\vec{v}_1 = 0$

$$\Rightarrow \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -2x - 2y = 0 \\ x + y = 0 \end{cases} \Rightarrow x + y = 0$$

pick $x = 1 \Rightarrow y = -1. \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solve $(\lambda_2 I_2 - A)\vec{v}_2 = 0$

$$\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow x + 2y = 0$$

pick $x = 1 \Rightarrow y = -\frac{1}{2} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$

Summarize: $\lambda_1 = 1, \longrightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 2 \longrightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$

Hence $\vec{x}_h = c_1 e^{t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\-\frac{1}{2}\end{bmatrix}$

Step 2. Find a particular soln to

$$\vec{x}\,' = \begin{bmatrix}3 & 2\\-1 & 0\end{bmatrix}\vec{x} + \begin{bmatrix}1\\2\end{bmatrix} \leftarrow \vec{f}$$

by using test function

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$$\vec{x}_p = \begin{bmatrix}a_1\\a_2\end{bmatrix}$$

plug $\vec{x}_p$ into the linear system

$$\vec{x}_p\,' = \begin{bmatrix}3 & 2\\-1 & 0\end{bmatrix}\vec{x}_p + \begin{bmatrix}1\\2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}3 & 2\\-1 & 0\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}3a_1+2a_2+1\\-a_1+2\end{bmatrix}$$

$$\Rightarrow \begin{cases}3a_1+2a_2+1=0\\-a_1+2=0\end{cases} \Rightarrow \begin{cases}a_1=2\\a_2=-\frac{7}{2}\end{cases}$$

$$\Rightarrow \quad \vec{x}_p = \begin{bmatrix} 2 \\ -\frac{7}{2} \end{bmatrix}$$

Step 3: Write down the general soln to the nonhomogeneous system.

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

$$= c_1 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 2 \\ -\frac{7}{2} \end{bmatrix}$$

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6. Use the method of variation of parameters to find the general soln to

$$\vec{x_p}' = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \vec{x} + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{f}}$$

A: Step 1: Find the general soln (or a fundamental soln set) to the associated homogeneous system.

$$\vec{x}' = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \vec{x}$$

we did this in Q5:

$$\lambda_1 = 1 \longrightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2 \longrightarrow v_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

$\Rightarrow$ a fundamental soln set:

$$\vec{x}_1 = e^{\lambda_1 t} \vec{v}_1 = e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

$$\vec{x}_2 = e^{\lambda_2 t} \vec{v}_2 = e^{2t} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ -\frac{1}{2} e^{2t} \end{bmatrix}$$

Step 2: Form a 2×2 matrix

$$X = [\vec{x}_1, \vec{x}_2] = \begin{bmatrix} e^t & e^{2t} \\ -e^t & -\frac{1}{2} e^{2t} \end{bmatrix}$$

Then compute X^{-1}

If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(X) = ad - bc \neq 0$

then $X^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\Rightarrow X^{-1} = \frac{1}{-\frac{1}{2}e^{3t} + e^{3t}} \begin{bmatrix} -\frac{1}{2}e^{2t} & -e^{2t} \\ e^{t} & e^{t} \end{bmatrix}$$

$$= \frac{2}{e^{3t}} \begin{bmatrix} -\frac{1}{2}e^{2t} & -e^{2t} \\ e^{t} & e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t} & -2e^{-t} \\ 2e^{-2t} & 2e^{-2t} \end{bmatrix}$$

Step 3: Compute $\vec{v} = \int X^{-1} \vec{f}\, dt$

Recall $\vec{f} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\Rightarrow X^{-1}\vec{f} = \begin{bmatrix} -e^{-t} - 4e^{-t} \\ 2e^{-2t} + 4e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} -5e^{-t} \\ 6e^{-2t} \end{bmatrix}$$

$$\Rightarrow \vec{v} = \int \begin{bmatrix} -5e^{-t} \\ 6e^{-2t} \end{bmatrix} dt$$

$$= \begin{bmatrix} \int -5e^{-t} dt \\ \int 6e^{-2t} dt \end{bmatrix}$$

$$= \begin{bmatrix} 5e^{-t} \\ -3e^{-2t} \end{bmatrix}$$

Then $\vec{x}_p = X\vec{v}$

$$= \begin{bmatrix} e^{t} & e^{2t} \\ -e^{t} & -\frac{1}{2}e^{2t} \end{bmatrix} \begin{bmatrix} 5e^{-t} \\ -3e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 5-3 \\ -5+\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{7}{2} \end{bmatrix}$$

Step 4: Write down the general soln to the given nonhomogeneous system.

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

$$= c_1 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 2 \\ -\frac{7}{2} \end{bmatrix}$$

Thanks, everyone!