Lecture 3

plan of this lecture :

- quick review of § 2.2 separable D.E
- § 2.3 1st order Linear D.E

Recall: § 2.2

Separable D.E

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

Ideas to solve separable D.E.

$$\frac{dy}{dx} = f(x)\, g(y). \qquad (2)$$

Step 1: Check whether $g(y) = 0$ gives a soln. to (2) (If $g(y)$ cannot be 0, then you don't need to do step 1)

Step 2: Suppose $g(y) \neq 0$.

$$(2) \Rightarrow \quad \frac{dy}{g(y)} = f(x)\, dx$$

Then integrate $\Rightarrow$

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

Calculus $\Rightarrow$ $\quad G(y) = F(x) + C$

Recall we did

E.g 1. solve $\frac{dy}{dx} = \frac{y-1}{x+3} = \underbrace{\frac{1}{x+3}}_{f(x)} \underbrace{(y-1)}_{g(y)}$ (3)

Step 1: Check whether $y-1=0$ gives a soln.

we got: $y=1$ is a soln.

Step 2: Assume $y-1 \neq 0$. Move terms about y to LHS. move terms about x to RHS then integrate $\Rightarrow$

we got:

$$\ln|y-1| = \ln|x+3| + C$$

We can rewrite it as

$$e^{\ln|y-1|} = e^{\ln|x+3|} e^{C}$$

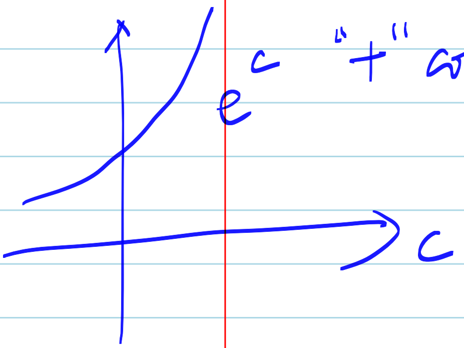
$$\Rightarrow |y-1| = e^{C}|x+3|$$

$$\Rightarrow y-1 = \pm e^{C}(x+3)$$

$|A| = |B| \Longleftrightarrow A = \pm B$

$\Rightarrow$

$y-1 = e^{c}(x+3)$; $\quad y-1 = -e^{c}(x+3)$



"+" constant $\qquad$ "−" constant

putting them together

$\Rightarrow \quad y-1 = A(x+3), \quad A \neq 0$

Q: Can "$A=0$"?

A: If $A=0$, the above becomes

$y-1=0 \Rightarrow y=1$ Yes!

Summarizing step 1, 2 $\Rightarrow$

$y-1 = A(x+3), \quad A \in \mathbb{R}$

$\Rightarrow \quad y = Ax + 3A + 1$

E.g2: $\dfrac{dy}{dx} = \dfrac{y^2+1}{x}$

A: $\dfrac{dy}{dx} = \underbrace{\dfrac{1}{x}}_{f(x)} \underbrace{(y^2+1)}_{g(y)}$

Step 1: check if $g(y)=0$ gives any soln.

But $g(y) = y^2+1 \geq 1$ cannot be zero.

$\Rightarrow$ No need to do step 1.

Step 2: Now $g(y) \neq 0$. Move y-terms to LHS.

x-terms to RHS

$$\Rightarrow \quad \frac{dy}{y^2+1} = \frac{1}{x}\,dx$$

Integrate $\Rightarrow \displaystyle\int \frac{dy}{y^2+1} = \int \frac{1}{x}\,dx$

Hint: $\displaystyle\int \frac{dy}{y^2+1} = \arctan y + C$, $\arctan y = \tan^{-1} y$

$\Rightarrow$

$$\arctan y = \ln|x| + C$$

$$\Rightarrow \quad y = \tan(\ln|x| + C)$$

§2.3 1st order Linear D.E

Recall: 1st order linear D.E is of the form:

$$a_1(x)\frac{dy}{dx} + a_0(x)\cdot y = b(x) \qquad (1)$$

Q: How to solve (1) ?

Let's first consider the following 1st order linear D.E :

$$x^2\frac{dy}{dx} + 2x\cdot y = 1 \qquad (*)$$

How to solve (*) ?

$y = y(x)$

Key observation:

$$\text{LHS of } (*) = x^2 y' + 2xy$$

$$= \frac{d}{dx}(\underbrace{x^2\cdot y}_{H(x,y)})$$

$$\Rightarrow \quad \frac{d}{dx}(x^2\cdot y) = 1$$

Integrate both sides $\Rightarrow$

$$\int \frac{d}{dx} F(x)\,dx = F(x) + C$$

$$\int \frac{d}{dx}(x^2 \cdot y)\,dx = \int 1\,dx$$

$$\Rightarrow \quad x^2 \cdot y = x + C$$

$$\Rightarrow \quad y = \frac{1}{x} + \frac{c}{x^2}$$

Key idea in solving $(*)$ is:

to recognize the LHS of $(*)$ as the derivative of some function, that is,

$$\text{LHS of } (*) = \frac{d}{dx} H(x, y)$$

for some function H.

Back to the Q: How to solve (1) ?

$$a_1(x)\frac{dy}{dx} + a_0(x)\, y = b(x) \qquad (1)$$

The difficulty is: the LHS of (1) might not be equal to the derivative of any function!

Key idea: To modify (1) to make the LHS of (1) = the derivative of some function

More precisely, we have the following algorithm to solve (1):

Q: Solve

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \qquad (1)$$

Algorithm:

Step 0: Make the eqn (1) in to the form:

$$\frac{dy}{dx} + p(x)y = Q(x) \qquad (2)$$

(Divide (1) by $a_1(x)$ $\Rightarrow$

$$\frac{dy}{dx} + \underbrace{\frac{a_0(x)}{a_1(x)}}_{p(x)} y = \underbrace{\frac{b(x)}{a_1(x)}}_{Q(x)}$$
)

Step 1: Calculate the following

$$\mu(x) = e^{\int p(x)dx}$$

— called the integrating factor of D.E (2)

Q: $\frac{d}{dx}\mu(x) = ?$

A: $\mu = e^{w}, \quad w = \int p(x)\,dx$

$\Rightarrow \mu' = e^{w}\cdot w' = e^{w}\cdot p(x) = \mu\cdot P$

Step 2: Multiply both sides of (2) by $\mu(x)$.

$\Rightarrow$

$$\underbrace{\mu(x)\frac{dy}{dx} + \mu(x)P(x)y}_{\parallel} = \mu(x)Q(x) \qquad (3)$$

$$\frac{d}{dx}(\mu(x)y) = \mu' y + \mu\cdot y'$$
$$= \mu\cdot P\cdot y + \mu\cdot y'$$

Step 3: Recognize the LHS of (3) as $\frac{d}{dx}[\mu(x)y(x)]$

(3) $\Rightarrow \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$

Integrate $\Rightarrow$

$$\mu(x)y = \int \mu(x)Q(x)\,dx$$

$$\Rightarrow \quad y = \frac{1}{\mu(x)}\int \mu(x)Q(x)\,dx$$

— often called the general soln to D.E (1) or (2)

E.g 1: Find the general soln to

$$\underbrace{\frac{1}{x}}_{a_1(x)}\frac{dy}{dx} - \frac{2y}{x^2} = \underbrace{x\cos x}_{b(x)} \qquad (4)$$

$a_2(x) = \frac{-2}{x^2}$

Step 0:

(4) $\Rightarrow$

$$\frac{dy}{dx} - \frac{2y}{x} = x^2\cos x \qquad (5)$$

$$\Rightarrow \begin{cases} P(x) = \frac{-2}{x} \\ Q(x) = x^2\cos x \end{cases}$$

Step 1: Compute $\mu(x)$

Remark: you can choose any value for C, and you always get the same final answer. We just choose the simpliest C: C = 0

$$\mu(x) = e^{\int p(x)dx} = e^{-2\int \frac{1}{x}dx}$$

$$= e^{-2(\ln|x| + C)}$$

choose $C=0$

$$= e^{-2(\ln|x|)}$$

$$= \left(e^{\ln|x|}\right)^{-2}$$

$$= (|x|)^{-2}$$

$$= \frac{1}{|x|^2} = \frac{1}{x^2}$$

$$\boxed{\begin{aligned} e^{AB} &= (e^A)^B \\ &= (e^B)^A \end{aligned}}$$

Step 2: Multiply the integrating factor $\mu(x)$

$\Rightarrow$

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \cos x \qquad (6)$$

Step 3. Recognize LHS of $(6) = \frac{d}{dx}\left(\frac{1}{x^2}y\right)$

$(6) \Rightarrow$

$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \cos x$$

Integrate $\Rightarrow$

$$\frac{1}{x^2}y = \int \cos x \, dx$$

$$= \sin x + C$$

$$\Rightarrow \quad y = x^2 \sin x + Cx^2$$

E.g 2.

① Find the general soln to

$$\frac{dy}{dx} - y = \frac{11}{8} e^{-\frac{x}{3}} \qquad (7)$$

② Solve the I.V.P

$$\begin{cases} \frac{dy}{dx} - y = \frac{11}{8} e^{-\frac{x}{3}} & \text{D.E} \\ y(0) = -1 & \text{Initial condition} \end{cases}$$

①

A:

Step 0: Not needed, already in the right form:

$$\begin{cases} p(x) = -1 \\ Q(x) = \frac{11}{8} e^{-\frac{x}{3}} \end{cases}$$

Step 1: Compute $\mu(x) = e^{\int p(x)dx}$

$$\Rightarrow \quad \mu(x) = e^{\int -1\, dx}$$

$$= e^{-x+c}$$

$$\underset{\text{Choose } c=0}{\rightarrow} \; = e^{-x}$$

Step 2. Mutiply (7) by $\mu(x)$ $\Rightarrow$

$$e^{-x}\frac{dy}{dx} - e^{-x}y = \frac{11}{8}e^{-\frac{4x}{3}} \qquad (8)$$

Step 3: Recognize

$$\text{LHS of } (8) = \frac{d}{dx}\left(e^{-x}y\right)$$

(8) $\Rightarrow$

$$\frac{d}{dx}\left(e^{-x}y\right) = \frac{11}{8}e^{-\frac{4}{3}x}$$

Integrate $\Rightarrow$

$$e^{-x}y = \underline{\int \frac{11}{8}e^{-\frac{4}{3}x}\,dx}$$

u-sub with $u = -\frac{4}{3}x$

$$e^{-x}y = -\frac{33}{32}e^{-\frac{4x}{3}} + C$$

Hence the general soln:

$$y = -\frac{33}{32}e^{-\frac{x}{3}} + Ce^{x} \qquad (9)$$

(2) Recall $y(0) = -1$.

Let $x = 0$, $y = -1$ in (9)

$$\Rightarrow \quad -1 = -\frac{33}{32} e^{0} + c e^{0}$$

$$= \frac{-33}{32} + c$$

$$\Rightarrow \quad c = \frac{1}{32}$$

Hence the soln to I.V.P:

$$y = -\frac{33}{32} e^{-\frac{x}{3}} + \frac{1}{32} e^{x}$$