Lecture 4

plan of Lecture 4:

- A remark on I.V.P (HW #1)
- § 2.4

Remark: when we work on an I.V.P

$$\begin{cases} F(x, y, \frac{\partial y}{\partial x}, \cdots, \frac{\partial^n y}{\partial x^n}) = 0 & \text{D.E} \\ y(x_0) = y_0, & \text{initial condition} \end{cases}$$

we can always assume that

$$\underbrace{(x, y) \text{ is close to the point } (x_0, y_0)}$$

This information sometimes can be useful.

problem 6 in #HW 1:

$$\frac{dy}{dx} + \frac{3y}{x} + 2 = 3x, \quad y(1) = 1$$

Note: This is a 1st order linear D.E

use the method in §2.3

Step 0:

$$\frac{dy}{dx} + \frac{3}{x} y = 3x - 2$$

$$\Rightarrow \begin{cases} P(x) = \frac{3}{x} \\ Q(x) = 3x - 2 \end{cases}$$

Step 1:

$$\mu(x) = e^{\int P(x)dx}$$

$$= e^{\int \frac{3}{x} dx}$$

$$= e^{3\ln|x|}$$

$$= |x|^3 = x^3$$

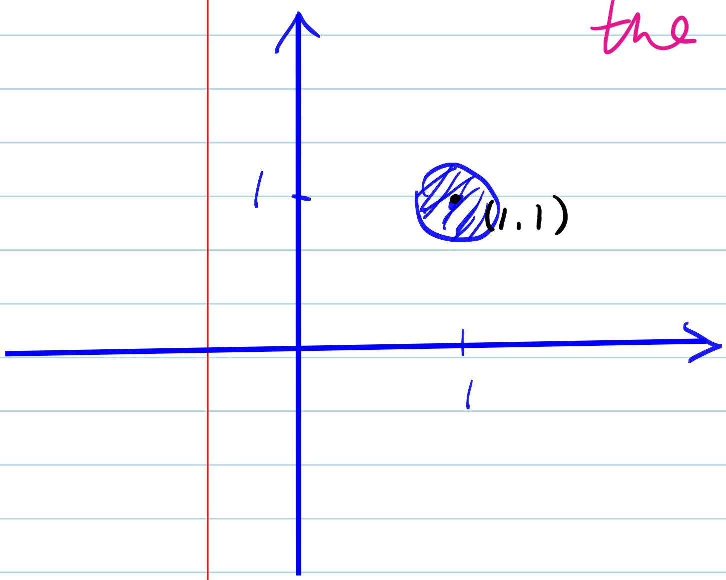
Q: Can we get rid of the absolute value sign $|\ |$?

A: Yes!

Because the initial condition

$$y(1) = 1 \Rightarrow (x_0, y_0) = (1, 1)$$

We can assume (x, y) is close to the given pt $(1, 1)$

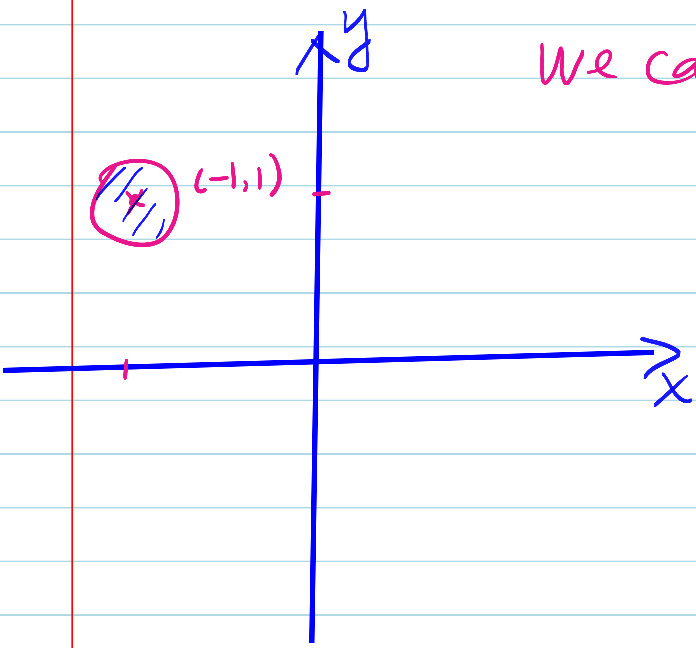


$$\Rightarrow \begin{cases} x \text{ is close to } 1 \\ y \text{ is close to } 1 \end{cases}$$

$$\Rightarrow \begin{cases} x > 0 \\ y > 0 \end{cases}$$

If the initial condition were

$$y(-1) = 1 \Rightarrow (x_0, y_0) = (-1, 1)$$



We can assume

$$\begin{cases} x \text{ is close to } -1 \\ y \text{ is close to } 1 \end{cases} \Rightarrow \begin{cases} x < 0 \\ y > 0 \end{cases}$$

§2.4 Exact equations

Def^n: ① By a differential form, we mean

$$M(x,y)\,dx + N(x,y)\,dy .$$

where $M(x,y)$, $N(x,y)$ are functions of x, y.

② Let $F(x,y)$ be a differentiable function. the total differential of F means

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Def^n: Let $M(x,y)dx + N(x,y)dy$ be a differential form. We say it is exact if there exists a differentiable function F such that

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF = M(x,y)\,dx + N(x,y)\,dy.$$

In this case, the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called an exact equation.

Remark: Let F be a differentiable function on a disk. Assume $dF = 0$ everywhere. Then F is a constant: $F = C$.

Why? This follows from Calculus:

Recall from

① For $F(x)$, if $\frac{dF}{dx} = F'(x) = 0$ everywhere

$\Rightarrow F$ is constant.

② For $F(x, y)$, if $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$

everywhere (i.e. $\nabla F = 0$ everywhere)

$\Rightarrow F$ is constant

If $dF = 0$ everywhere

Recall $$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

$$\Rightarrow \begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \end{cases} \text{ everywhere}$$

$\Rightarrow F$ is constant: $F = C$

E.g.: Consider

$$\frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y}$$

We convert it into the form "$Mdx+Ndy=0$"

$$(2xy^2+1)\,dx + (2x^2y)\,dy = 0 . \quad (1)$$

Then
$$\begin{cases} M(x,y) = 2xy^2+1 \\ N(x,y) = 2x^2y \end{cases}$$

Let $F = x^2y^2 + x$. Then

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$
$$= (2xy^2+1)dx + (2x^2y)dy$$

Hence
$$\text{LHS of } (1) = dF$$

Thus (1) becomes $dF = 0$

$\Rightarrow \quad F = C$

$\Rightarrow \quad x^2y^2 + x = C$

It is a soln to the original D.E

This gives some ideas to solve exact equations:

Assume $M(x,y)dx + N(x,y)dy = 0 \quad (*)$ is exact.

Suppose we can find $F(x,y)$ such that

$$dF = M(x,y)\,dx + N(x,y)\,dy = 0$$

Then the soln of $(*)$ is $F = C$

But two questions:

Q1: How to check whether (*) is exact?

Q2. Once we know (*) is exact, how to find the expression of F?

Answer to Q1:

Thm (Test for exactness)

The differential form $M(x,y)dx + N(x,y)dy$ is exact "$\Longleftrightarrow$" $\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$.

(Note: "$\Longleftrightarrow$" means "is equivalent to" or "if and only if")

Pf: We will skip the pf and give some intuition behind the pf.

Suppose there exists F such that

$$dF = \underset{\parallel}{M}(x,y)\,dx + \underset{\parallel}{N}(x,y)\,dy$$

$$\left(\text{Recall } dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \right)$$

$$\Rightarrow \quad \begin{cases} \dfrac{\partial F}{\partial x} = M(x,y) & ① \\ \dfrac{\partial F}{\partial y} = N(x,y) & ② \end{cases}$$

Then

Apply $\frac{\partial}{\partial y}$ to ① $\Longrightarrow$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial M}{\partial y} \quad ;$$

Apply $\frac{\partial}{\partial x}$ to ② $\Longrightarrow$

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

From Calculus, we know $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial X}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

E.g1: Is $2x\,dx + 2y\,dy$ exact?

A: Note $\begin{cases} M(x,y) = 2x \\ N(x,y) = 2y \end{cases}$

$$\Rightarrow \begin{cases} \frac{\partial M}{\partial y} = 0 \\ \frac{\partial N}{\partial x} = 0 \end{cases} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ exact!

E.g2: Is $x\,dx + x^2\,dy$ exact?

A: Note $\begin{cases} M(x,y) = x \\ N(x,y) = x^2 \end{cases}$

$$\Rightarrow \begin{cases} \frac{\partial M}{\partial y} = 0 \\ \frac{\partial N}{\partial x} = 2x \end{cases} \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ NOT exact!

Answer to Q2: once we know it is exact, how to find F. ?

E.g. Solve $\frac{dy}{dx} = -\frac{2x+y}{2y+x}$

A:

Step 0: Reduce to the form $M dx + N dy = 0$

$$(2x+y)dx + (2y+x)dy = 0$$

Step 1: Check the exactness.

Note $\begin{cases} M(x,y) = 2x+y \\ N(x,y) = 2y+x \end{cases}$

$$\Rightarrow \begin{cases} \frac{\partial M}{\partial y} = 1 \\ \frac{\partial N}{\partial x} = 1 \end{cases} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{exact!}$$

Next we will find F s.t

$$\begin{cases} \dfrac{\partial F}{\partial x} = 2x+y \leftarrow M & ① \\ \dfrac{\partial F}{\partial y} = 2y+x \rightarrow N & ② \end{cases}$$

Step 2: Regard y as a constant and integrate ① with respect to x

$$F = \int \frac{\partial F}{\partial x}\, dx = \int (2x+y)\, dx$$

$$= x^2 + yx + \boxed{g(y)} \leftarrow \text{constant term}$$

$$\Rightarrow \quad F = x^2 + yx + g(y) \qquad (3)$$

Step 3. Substitute what you get in step 2 (i.e., (3)) into ②

$$\Rightarrow x + g'(y) = \frac{\partial F}{\partial y} = 2y + x$$

$$\Rightarrow \quad g'(y) = 2y \Rightarrow g = \int 2y\,dy$$
$$= y^2 + C$$

choose $C = 0 \Rightarrow g(y) = y^2$

Hence the soln to the D.E is.

$F = C$, that is

$$x^2 + yx + y^2 = C$$

E.g: Solve

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0.$$

A:

Step 0. Already in the form "$M\,dx + N\,dy = 0$"

Step 1: check the exactness.

Note $\begin{cases} M(x,y) = 2xy - \sec^2 x \\ N(x,y) = x^2 + 2y \end{cases}$

$$\Rightarrow \begin{cases} \dfrac{\partial M}{\partial y} = 2x \\ \dfrac{\partial N}{\partial x} = 2x \end{cases}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{exact!}$$

We will find F s.t

$$\begin{cases} \dfrac{\partial F}{\partial x} = M = 2xy - \sec^2 x & (1) \\ \dfrac{\partial F}{\partial y} = N = x^2 + 2y & (2) \end{cases}$$

Step 2: Integrate (1)

$$F = \int \frac{\partial F}{\partial x}\, dx = \int (2xy - \sec^2 x)\, dx$$

$$= x^2 y - \tan x + g(y)$$

Hint:

$\sec x = \dfrac{1}{\cos x}$

$\int \sec^2 x\, dx = \tan x + C$

Step 3: plug the above into (2)

$\Rightarrow$

$$x^2 + g'(y) = \frac{\partial F}{\partial y} = x^2 + 2y$$

$$\Rightarrow \quad g'(y) = 2y \Rightarrow g(y) = \int 2y \, dy$$
$$= y^2 + C$$

Choose $C = 0 \Rightarrow g(y) = y^2$

$$\Rightarrow F = x^2 y - \tan x + y^2$$

Hence the soln to the D.E is

$$x^2 y - \tan x + y^2 = C$$