Lecture 5

Reminder:

1. paper-and-pen HW1 is due April 9th
2. Matlab HW1 is due April 16th

Plan of Lecture 5:

- very quick review of §2.4
- §2.5

Review of §2.4

Test

- "$M(x,y)dx + N(x,y)dy = 0$" is exact

$$\Longleftrightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- If $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$

$$= M(x,y)\,dx + N(x,y)\,dy$$

then "$M dx + N dy = 0$" has an implicit solution $F(x,y) = C$

- when "$Mdx + Ndy = 0$" is exact, how to find F s.t $dF = Mdx + Ndy$?

we need to find F s.t

$$\begin{cases} \dfrac{\partial F}{\partial x} = M & ① \\ \dfrac{\partial F}{\partial y} = N & ② \end{cases}$$

Step 1: Integrate ① w.r.t x by regarding y as a constant

Step 2: substitute what we get in st 1 into ②

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§ 2.5 Integrating factor

Q: What if "$Mdx + Ndy = 0$" is NOT exact? Can we still solve it?

A: For some "nice" non-exact equation. Yes!

Indeed, for some non-exact equation

$$M\,dx + N\,dy = 0, \qquad (1)$$

We can "manually" make it exact by multiplying some function μ. That is,

$$\underbrace{\mu(x,y)\,M(x,y)}_{M_1}dx + \underbrace{\mu(x,y)\,N(x,y)}_{N_1}dy = 0 \qquad (2)$$

is exact

In this case, μ is called an integrating factor of (1)

Q: what the relation between "integrating factor" here and "integrating factor" in §2.3?

A: we will discuss at the end.

In short, they are different but highly similar in spirit.

First let's consider the following Q:

Q: How to find $\mu(x,y)$ in (2)?

A: Let $\begin{cases} M_1 = \mu M \\ N_1 = \mu N \end{cases}$

(2) is exact $\Longleftrightarrow$ $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$

This means

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N].$$

$$\Rightarrow \partial_y \mu M + \mu \partial_y M = \partial_x \mu N + \mu \partial_x N$$

Notations:

$\partial_x \mu = \dfrac{\partial \mu}{\partial x}$

$\partial_y \mu = \dfrac{\partial \mu}{\partial y}$

$\partial_y M = \dfrac{\partial M}{\partial y}$

$\partial_x N = \dfrac{\partial N}{\partial x}$

$$\Rightarrow M \partial_y \mu - N \partial_x \mu = (\partial_x N - \partial_y M)\mu \qquad (3)$$

In general, it is very difficult to solve (3)

But in some exceptional cases, we can solve it.

(I) Suppose we can make $\partial_y \mu = 0$. Then (3) $\Rightarrow$

$$-N \partial_x \mu = (\partial_x N - \partial_y M)\mu$$

$$\Rightarrow \quad \partial_x \mu = \frac{1}{N}(\partial_y M - \partial_x N)\mu \qquad (4)$$

Suppose " $\frac{1}{N}(\partial_y M - \partial_x N)$ " only depends on x.

Call it $f(x)$. Then (4) $\Rightarrow$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{\mu}\partial_x \mu = f(x)$$

$$\Rightarrow \quad \frac{1}{\mu}d\mu = f(x)\,dx$$

$$\Rightarrow \quad \int \frac{1}{\mu}d\mu = \int f(x)\,dx$$

only depends on x

$$\Rightarrow \quad \ln|\mu| = \int f(x)\,dx$$

$$\Rightarrow \quad |\mu| = e^{\int f(x)\,dx} \Rightarrow \mu = \pm e^{\int f(x)dx}$$

both of " $\pm$ " will work. We always choose the "+" one. $\Rightarrow \mu = e^{\int f(x)dx}$ depends only on x

Summarize:

If $\frac{1}{N}(\partial_y M - \partial_x N)$ depends on x, then only

$\mu = e^{\int f(x)dx}$ works!

(II) Suppose we can make $\partial_x \mu = 0$. Then

(3) becomes

$$M\, \partial_y \mu = (\partial_x N - \partial_y M)\mu$$

$$\Rightarrow \quad \partial_y \mu = \frac{1}{M}(\partial_x N - \partial_y M)\mu \qquad (5)$$

Suppose $\frac{1}{M}(\partial_x N - \partial_y M)$ depends only on y,

call it $g(y)$. $(5) \Rightarrow$

$$\frac{1}{\mu}\frac{d\mu}{dy} = \frac{1}{\mu}\partial_y \mu = g(y)$$

$$\Rightarrow \quad \frac{1}{\mu} d\mu = g(y)\,dy$$

$$\Rightarrow \quad \int \frac{1}{\mu}\, d\mu = \int g(y)\, dy$$

$$\Rightarrow \quad \ln|\mu| = \int g(y)\, dy \Rightarrow$$

$$\mu = \pm e^{\int g(y)\, dy}$$

we normally take the "+" one $\Rightarrow$

$$\mu = e^{\int g(y)\,dy} = e^{\int \frac{1}{M}(\partial_x N - \partial_y M)\,dy}$$

depends only on y

Note this μ is a function depending only on y! $\Rightarrow \frac{\partial \mu}{\partial x} = 0.$

Summarize: If $\frac{1}{M}(\partial_x N - \partial_y M)$ depends only on y, then $\mu = e^{\int g\, dy}$ works!

We summarize the above to the following theorem:

Thm. Given $Mdx + Ndy = 0$ $(*)$

(I) If $\frac{1}{N}(\partial_y M - \partial_x N)$ depends only on x

(that is, does not depend on y), then

$$\mu = \mu(x) = e^{\int \frac{1}{N}(\partial_y M - \partial_x N)\,dx}$$

is an integrating factor of $(*)$

(means "$\mu M dx + \mu N dy = 0$" is exact)

(II) If $\frac{1}{M}(\partial_x N - \partial_y M)$ only depends on

y (that is, does not depend on x), then

$$\mu = \mu(y) = e^{\int \frac{1}{M}(\partial_x N - \partial_y M)\,dy}$$

is an integrating factor of $(*)$

(means "$\mu M dx + \mu N dy = 0$" is exact)

Remark: only use the above theorem when

$(*)$ is not exact.

E.g. Solve

$$\underbrace{(2x^2+y)}_{M}dx + \underbrace{(x^2y-x)}_{N}dy = 0. \quad (6)$$

A: Note $\begin{cases} M = 2x^2+y \\ N = x^2y - x \end{cases}$

Step 1: check the exactness of (6).

$$\begin{cases} \frac{\partial M}{\partial y} = 1 \\ \frac{\partial N}{\partial x} = 2xy - 1 \end{cases}$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{not-exact!}$$

Step 2. Can we use the theorem?

Need to check!

(I) Compute

$$\frac{1}{N}(\partial_y M - \partial_x N)$$

$$= \frac{1-(2xy-1)}{x^2y-x} = \frac{-2(xy-1)}{x(xy-1)} = \frac{-2}{x}$$

It only depends on x!

Hence, yes, we can use Theorem (I).

The integrating factor

$$\mu(x) = e^{\int -\frac{2}{x}dx}$$

$$= e^{-2\ln|x| + c}$$ choose $c=0$

$$= \frac{1}{|x|^2} = \frac{1}{x^2} = x^{-2}$$

Step 3. Multiply both sides by $\mu(x)$

$$x^{-2}(2x^2+y)dx + x^{-2}(x^2y - x)dy = 0$$

$$\Rightarrow \quad (2+\frac{y}{x^2})dx + (y - \frac{1}{x})dy = 0$$

This new D.E is exact! ← use last lecture

Step 4. We next find F s.t

$$\begin{cases} \frac{\partial F}{\partial x} = 2 + \frac{y}{x^2} & (1) \\ \frac{\partial F}{\partial y} = y - \frac{1}{x} & (2) \end{cases}$$

Integrating ① and regarding y as a constant

$$\Rightarrow \quad F = \int \frac{\partial F}{\partial x}\,dx = \int \left(2 + \frac{y}{x^2}\right) dx$$

$$= 2x - \frac{y}{x} + \boxed{g(y)}$$ constant term

Substitute into ② $\Rightarrow$

$$-\frac{1}{x} + g'(y) = \frac{\partial F}{\partial y} = y - x^{-1}$$

$$\Rightarrow \quad g'(y) = y \quad \Rightarrow \quad g(y) = \frac{1}{2}y^2$$

Hence $F = 2x - \frac{y}{x} + \frac{1}{2}y^2$

the sol/n to the D.E is $(F = C)$

$$2x - \frac{y}{x} + \frac{1}{2}y^2 = C.$$

§ 2.5

Q: what the relation between "integrating factor" here and "integrating factor" in § 2.3?

Recall "integrating factor" in § 2.3:

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

The integrating factor $\mu(x) = e^{\int P(x)dx}$

Sometimes you can both of the "integrating factor" methods to solve a D.E.

E.g Solve

$$\frac{dy}{dx} + \underbrace{\frac{3}{x}}_{P(x)} y = \underbrace{2}_{Q(x)}, \quad x>0. \qquad (7)$$

Way 1: (Use § 2.3)

Note $\begin{cases} P(x) = \frac{3}{x} \\ Q(x) = 2 \end{cases}$

$$\Rightarrow \quad \mu = e^{\int P(x)dx} = \underbrace{e^{\int \frac{3}{x}dx}}_{= e^{3\ln|x|}} = x^3$$

Multiply μ to both sides $\Rightarrow$

$$x^3 \frac{dy}{dx} + 3x^2 y = 2x^3$$

$$\Rightarrow \quad \frac{d}{dx}(x^3 y) = 2x^3$$

$$\Rightarrow \quad x^3 y = \int 2x^3 dx = \frac{1}{2} x^4 + C$$

$$\Rightarrow \quad y = \frac{1}{2} x + \frac{C}{x^3}$$

Way 2 (use today's lecture)

Reduce (7) to the form "$M dx + N dy = 0$"

$$(7) \Rightarrow \quad dy = \left(2 - \frac{3y}{x}\right) dx$$

$$\Rightarrow \quad \left(\frac{3y}{x} - 2\right) dx + dy = 0 \qquad (8)$$

$$\Rightarrow \begin{cases} M = \frac{3y}{x} - 2 \\ N = 1 \end{cases}$$

Step 1: Check exactness

$$\begin{cases} \frac{\partial M}{\partial y} = \frac{3}{x} \\ \frac{\partial N}{\partial x} = 0 \end{cases} \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ not exact!

Step 2. Try to use the theorem.

Check which case it is ((I) or (II))

Check (I): Compute

$$\frac{1}{N}(\partial_y M - \partial_x N)$$

$$= \frac{1}{1}\left(\frac{3}{x} - 0\right) = \frac{3}{x}$$

It only depends on x!

Use Case (I).

$$\mu = e^{\int \frac{3}{x} dx} = e^{3\ln|x|} = |x|^3 = x^3$$

Multiply (8) by $\mu \Rightarrow$

$$(3x^2y - 2x^3)\,dx + x^3\,dy = 0$$

This new D.E is exact!

We will find F s.t

$$\begin{cases} \frac{\partial F}{\partial x} = 3x^2y - 2x^3 & ① \\ \frac{\partial F}{\partial y} = x^3 & ② \end{cases}$$

Integrating ① $\Rightarrow$

$$F = \int (3x^2y - 2x^3)\,dx$$

$$= x^3y - \frac{1}{2}x^4 + g(y)$$

Substitute F into ②

$$x^3 + g' = \frac{\partial F}{\partial y} = x^3$$

$$\Rightarrow g' = 0 \quad \Rightarrow g = C$$

choose $C = 0 \Rightarrow g = 0$

$$\Rightarrow F = x^3y - \frac{1}{2}x^4$$

The soln to the D.E is

$$x^3 y - \frac{1}{2} x^4 = C$$