Lecture 6

plan of Lecture 6

- § 4.2

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§ 4.2 Homogeneneous Linear Equation (2nd order)

<u>Def^n</u>: By a linear 2nd order "constant coefficient" D.E, we mean

$$ay'' + by' + cy = f(x) \quad (1)$$

Here a, b, c are constants and $a \neq 0$.

- If $f = 0$, then (1) becomes

$$ay'' + by' + cy = 0.$$

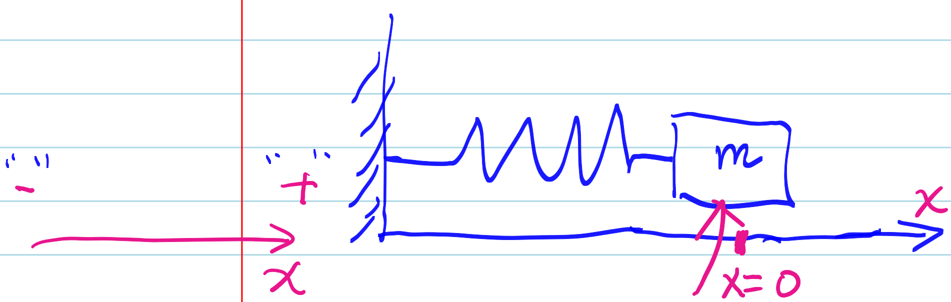
——— we say it is homogeneous

• If $f \neq 0$

—— we say it is nonhomogeneous

Remark: There are many examples of linear 2nd order constant coefficient D.Es in real life.

E.g. a mass on a spring



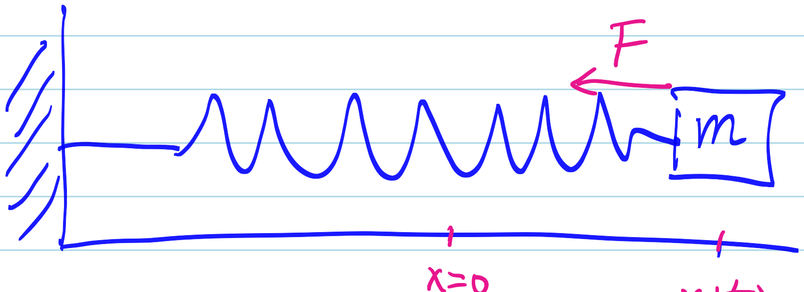
m: mass

k: spring costant

$x=0$: rest position of the spring

x: the position of the mass

At time t



$x(t)$: position of the mass at t.

Newton's second law:

$$F = m a$$

force mass accelaration

Hooke's law:

force exerted by the spring $= k \cdot x$

$$\Rightarrow \quad F = -kx$$

Remark:
when $x > 0$
F is negative pointing

Note $a = \frac{d^2x}{dt^2} = x''$

$$F = ma \Rightarrow -kx = mx''$$

or $mx'' + kx = 0$

Q: How to solve

$$ay'' + by' + cy = f(x) \ ?$$

Today we will consider the easier case where $f = 0$, i.e. the homogeneous case:

$$ay'' + by' + cy = 0 \qquad (*)$$

Let's first discuss some properties of $(*)$.

properties

① If $y_1(x)$ is a sol/n of $(*)$, C_1 is constant, then $C_1 y_1(x)$ is also a sol/n of $(*)$.

② If $y_1(x)$ and $y_2(x)$ are both sol/ns of $(*)$, then

$y_1(x) + y_2(x)$ is also a sol/n of $(*)$.

③ If $y_1(x)$ and $y_2(x)$ are solns of $(*)$, c_1, c_2 are constants, then

$c_1 y_1(x) + c_2 y_2(x)$ is a soln of $(*)$.

Pf: ①

Since y_1 is a soln $\Rightarrow$

$$a y_1'' + b y_1' + c y_1 = 0$$

Let's check $c_1 y_1$.

$$LHS = a(c_1 y_1)'' + b(c_1 y_1)' + c(c_1 y_1)$$

$$= c_1 (a y_1'' + b y_1' + c y_1) = 0$$

② Since y_1, y_2 are solns, $\Rightarrow$

$$\begin{cases} a y_1'' + b y_1' + c y_1 = 0 \\ a y_2'' + b y_2' + c y_2 = 0 \end{cases}$$

$y_1 + y_2$

$$\Rightarrow \quad a(y_1+y_2)'' + b(y_1+y_2)' + c(y_1+y_2) = 0$$

EX

$\Rightarrow$ $y_1 + y_2$ is a soln!

③ Exercise!

$y_1 = e^x$

$y_1' = e^x$

$y_1'' = e^x$

E.g. $y_1(x) = e^x$ is a soln of $y'' = y$.

Then $5e^x$ is also a soln

$10e^x$ is also a soln.

every $c_1 e^x$ is also a soln for $c_1 \in \mathbb{R}$

Let's come back to

Q: How to solve $ay'' + by' + cy = 0$? $(*)$

A: Let's first try simply ones:

Find out one soln of:

① $y'' - y = 0$ $(a=1, b=0, c=-1)$

$\Leftrightarrow$ $y'' = y$

one soln: $y_1 = e^x$

② $y'' - 4y = 0 \iff y'' = 4y$

One soln: $y_1 = e^{2x}$

$$\boxed{\begin{aligned} y_1' &= 2e^{2x} \\ y_1'' &= 2 \cdot 2 e^{2x} \\ &= 4e^{2x} \\ &= 4y_1 \end{aligned}}$$

③ $y'' - 9y = 0 \iff y'' = 9y$

One soln: $y_1 = e^{3x}$

④ In general,

$$y'' = A^2 y. \quad A \in \mathbb{R}$$

has a soln: $y_1 = e^{Ax}$

Can we use the same idea to solve

$$ay'' + by' + cy = 0 \quad (*) \quad ?$$

$(a \neq 0)$

Yes! The idea is: Try $e^{\lambda x}$.

λ is a constant to be determined!

$y' = \lambda e^{\lambda x}$
$y'' = \lambda^2 e^{\lambda x}$

That is, suppose $y = e^{\lambda x}$ solves

$$ay'' + by' + cy = 0 \quad \text{Then}$$

$$a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + c(e^{\lambda x}) = 0$$

$$\Rightarrow \quad e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$$

Q: How to make the above hold?

A: $e^{\lambda x}$ cannot be zero, we thus need

$$a\lambda^2 + b\lambda + c = 0 \, !$$

Def^n: $a\lambda^2 + b\lambda + c = 0$ is called the characteristic eqn of $(*)$

E.g $\quad y'' - 5y' + 6y = 0 \qquad (2)$

Assume $y = e^{\lambda x}$ is a sol'n. we need

$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow$$

$$(\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3.$$

Thus we obtain two sol'ns: $e^{2x}, \; e^{3x}$

(you can verify they are both sol'ns!)

Check: ① e^{2x}

$$(e^{2x})'' - 5(e^{2x})' + 6e^{2x}$$
$$= 4e^{2x} - 10e^{2x} + 6e^{2x} = 0$$

② e^{3x} Exercise.

By properties: for any $c_1, c_2 \in \mathbb{R}$,

$c_1 e^{2x} + c_2 e^{3x}$ is a soln !

Idea: In general, to solve

$$ay'' + by' + cy = 0, \quad a \neq 0 \qquad (1)$$

we need to solve $a\lambda^2 + b\lambda + c = 0$.

If λ satisfies $a\lambda^2 + b\lambda + c = 0$, then

$e^{\lambda x}$ is a soln of (1).

Recall quadratic formula:

$$a\lambda^2 + b\lambda + c = 0, \quad a \neq 0$$

$$\Rightarrow \quad \lambda = \frac{-b \pm \sqrt{\Delta}}{2a}. \qquad \Delta = b^2 - 4ac$$

determinant

Three cases:

(I) $\Delta > 0$, two distint real roots, $\lambda_1 \neq \lambda_2$

(II) $\Delta = 0$, one repeated real root, $\lambda_1 = \lambda_2$

(III) $\Delta < 0$, no real roots.

$$ay'' + by' + cy = 0 \qquad (*)$$

Now two questions.

Q1: How to find all solns to $(*)$?

Q2. Are the treatments for cases (I), (II), (III) the same?

For example, in (III), there are no roots!

What should we do?

A to Q1:

<u>Def</u>^{n}: Let $y_1(x)$ and $y_2(x)$ be two functions on an interval I.

- we say y_1 and y_2 are "Linearly dependent" ("L.D.") if one of them is a constant multiply of the other ($y_1 = k y_2$ or $y_2 = k y_1$)
- we say y_1 and y_2 are "Linearly independent" ("L.I.") if neither of them is a constant multiply to the other.

How to check whether y_1, y_2 are L.I or L.D?

Thm: Define the wronskian of y_1, y_2 to be

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

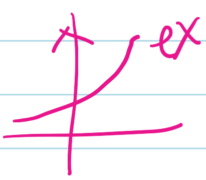
Then y_1, y_2 are L.D. on an interval I

"$\Longleftrightarrow$" $W(y_1, y_2) = 0$ everywhere on I.

This means

(1). If $W(y_1, y_2)$ is everywhere 0, $\Rightarrow$ y_1, y_2 are L.D.

(2) If $W(y_1, y_2)$ is NOT everywhere $0 \Rightarrow$ y_1, y_2 are L.I



E.g 1: Let $y_1 = x+1$, $y_2 = e^x$

① verify $W(y_1, y_2) = xe^x$ Exercise

$$\overset{\shortparallel}{y_1 y_2' - y_2 y_1'}$$

$e^x \neq 0$ everywhere

② Since $W(y_1, y_2) = 0$ only at one point $x=0$

$\Rightarrow$ y_1, y_2 is L.I on $\mathbb{R}$.

E.g 2: Let $y_1 = x$, $y_2 = 2x$

$\Rightarrow$ y_1, y_2 are L.D. ($y_2 = k y_1$, with $k=2$)

Can we check by wronskian?

Yes, $W(x, 2x) = y_1 y_2' - y_2 y_1'$

$= x \cdot 2 - 2x \cdot 1 = 0$ everywhere

Remark:

(1). If $\lambda_1 \neq \lambda_2$, then $e^{\lambda_1 x}, e^{\lambda_2 x}$ are L.I

(2) If $\lambda_1 = \lambda_2$, then $e^{\lambda_1 x}, e^{\lambda_2 x}$ are L.D.

Why? Use wronskian

$$W(y_1, y_2) = e^{\lambda_1 x}(e^{\lambda_2 x})' - (e^{\lambda_2 x}) \cdot (e^{\lambda_1 x})'$$

$$= (\lambda_2 - \lambda_1) e^{\lambda_1 x} e^{\lambda_2 x} \begin{cases} = 0 & \text{if } \lambda_1 = \lambda_2 \\ \neq 0 & \text{if } \lambda_1 \neq \lambda_2 \end{cases}$$

Thm: Given

$$ay'' + by' + cy = 0 \qquad (1)$$

If y_1, y_2 are two "L.I" solns of (1)

then "$c_1 y_1 + c_2 y_2$" gives all solns of (1)

— called the general solns of (1)

where $c_1, c_2 \in \mathbb{R}$

This means, every soln of (1) can be obtained from the form "$c_1 y_1 + c_2 y_2$".

Remark: By the Thm, to find the general solns of (1), we just need to find two L.I solns of (1).

Now consider the case $\Delta = b^2 - 4ac > 0$

Then "$a\lambda^2 + b\lambda + c = 0$"

has two distinct solns: λ_1, λ_2. ($\lambda_1 \neq \lambda_2$).

$y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are both solns of

$$ay'' + by' + cy = 0 \qquad (1)$$

And y_1, and y_2 are "L.I."!!!

By the Thm,

$$c_1 y_1 + c_2 y_2 = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

gives all the solns of (1)

E.g. Consider $y'' - 5y' + 6y = 0$ (2)

Step 1: Solve the characteristic eqn:

$\Delta = b^2 - 4ac$
$= 25 - 4 \cdot 1 \cdot 6$
$= 1 > 0$

$$"\lambda^2 - 5\lambda + 6 = 0"$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$\Rightarrow$ Two distinct roots: $\lambda_1 = 2, \lambda_2 = 3$.

Step 2: If there are two distinct solns λ_1, λ_2 in step 1, then there are two L.I. solns to (3): $e^{\lambda_1 x}$, $e^{\lambda_2 x}$

$$y_1(x) = e^{2x}, \quad y_2(x) = e^{3x}.$$

The general soln:

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 e^{2x} + c_2 e^{3x}$$

Now what if $\Delta = 0$. ?

E.g $\quad y'' + 4y' + 4y = 0$

characteristic eqn: "$\lambda^2 + 4\lambda + 4 = 0$"

$\Rightarrow \Delta = 4^2 - 4 \times 4 = 0.$

It has only one repeated root

$$\lambda = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-4}{2} = -2.$$

we can get in this way only "one" soln.

"e^{-2x}"

How to get another soln ?

Thm: If the characteristic eqn

$$a\lambda^2 + b\lambda + c = 0,$$

has only one repeated root λ_0.

Then $e^{\lambda_0 x}$, $xe^{\lambda_0 x}$ are two "L.I" solns, and the general solns are

$$c_1 e^{\lambda_0 x} + c_2 x e^{\lambda_0 x}.$$

where $c_1, c_2 \in \mathbb{R}$.

check: $W(e^{\lambda_0 x}, x e^{\lambda_0 x}) \neq 0$

E.g.: Recall

$$y'' + 4y' + 4y = 0 \qquad (3)$$

The characteristic eqn

$$\lambda^2 + 4\lambda + 4 = 0$$

has a repeated root $\lambda = -2$.

Then (3) has two "L.I" solns.

$$y_1 = e^{-2x}, \quad y_2 = x e^{-2x}$$

Exercise: check y_2 is a soln of (3)!

$$y_2' = e^{-2x} - 2x e^{-2x}$$

$$y_2'' = -2e^{-2x} - 2e^{-2x} + 4x e^{-2x}$$

$$\Rightarrow \quad y_2'' + 4y_2' + 4y_2 = \cdots\cdots = 0$$

Ex.

The general soln of (3) is

$$y = C_1 e^{-2x} + C_2 x e^{-2x}.$$