Lecture 9

plan of lecture 9:

§ 4.4 undetermined coefficients

We are now familiar with solving homogeneous 2nd order linear D.E with constant coefficients:

$$ay'' + by' + cy = 0 \qquad (1)$$

Today we discuss the more complicated case —— "nonhomogeneous" case, that is, the RHS of (1) $= f(x)$, which is not 0:

$$ay'' + by' + cy = f(x) \qquad (*)$$

First we make a definition

- Any one solution of a D.E is called a particular solution of the D.E (comparing to the 'general solution'.)

E.g.:

$$y'' + 3y' + 2y = 0$$

The characteristic eqn $\lambda^2 + 3\lambda + 2 = 0$

$\Rightarrow (\lambda+1)(\lambda+2) = 0$

$\Rightarrow \lambda_1 = -1, \ \lambda_2 = -2$

General solution:

$$C_1 e^{-x} + C_2 e^{-2x}, \quad C_1, C_2 \in \mathbb{R}$$

Then

$C_1, C_2 = 1 \longrightarrow$ $e^{-x} + e^{-2x}$ is a particular solution;

$C_1 = \frac{1}{2}, C_2 = 2 \rightarrow$ $\frac{1}{2} e^{-x} + 2e^{-2x}$ is a particular soln;

e^{-x} is a particular soln; etc.

E.g. Find a particular solution to

$$y'' + 3y' + 2y = 3x . \quad (2)$$

A: Hint: try $y = ax + b$, where a, b are constants whose values are to be determined

By the hint, try $y = ax + b$. In the case,

$$y'' = 0, \quad y' = a.$$

Suppose y solves the D.E (2), then

$$LHS = \underbrace{0}_{y''} + \underbrace{3a}_{3y'} + \underbrace{2(ax+b)}_{2y} = 3x = RHS$$

$$\Rightarrow \quad 2ax + (3a + 2b) = 3x$$

Compare two sides of the above eqn

x-term: $2ax = 3x$ ①

Constant: $3a + 2b = 0$ ②

$$\Rightarrow \begin{cases} 2a = 3 \\ 3a + 2b = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = \frac{3}{2} \\ b = -\frac{9}{4} \end{cases}$$

Hence $y_p = \frac{3}{2}x - \frac{9}{4}$ solves the D.E

and thus is a particular soln.

Remark: The above method is called "undetermined coefficient method".

The key point in the above E.g is that the hint gives the form $y = ax + b$.

E.g 1: Find a particular soln to

$$y'' - 4y' + 4y = 5e^{-3x}. \qquad (\Delta)$$

Hint: Try $y = Ae^{-3x}$, A to be determined

Let $y = Ae^{-3x}$. Then

$$y' = -3Ae^{-3x}, \quad y'' = 9Ae^{-3x}$$

Then $y'' - 4y' + 4y = 5e^{-3x} \Rightarrow$

$$\underbrace{9Ae^{-3x}}_{y''} + \underbrace{12Ae^{-3x}}_{-4y'} + \underbrace{4Ae^{-3x}}_{4y} = \underbrace{5e^{-3x}}$$

$$\Rightarrow \quad 25A\, e^{-3x} = 5e^{-3x}$$

$$\Rightarrow \quad 25A = 5$$

$$\Rightarrow \quad A = \frac{1}{5}$$

Hence $y_p = \frac{1}{5}e^{-3x}$ is a particular soln.

E.g 2: Find a particular soln of

$$y'' + 4y' + 3y = 5e^{-3x} \qquad (3)$$

Can we still try $y = Ae^{-3x}$. ?

If $y = Ae^{-3x} \Rightarrow y' = -3Ae^{-3x}, \; y'' = 9Ae^{-3x}$

$$\Rightarrow \; y'' + 4y' + 3y = 9Ae^{-3x} - 12Ae^{-3x} + \underline{3Ae^{-3x}}$$

LHS

$$= 0 \neq 5e^{-3x} = RHS$$

Cannot make them equal!

Hence "$y = Ae^{-3x}$" does not work!

Hint: try $y = Axe^{-3x}$!

$$\Rightarrow y' = Ae^{-3x} - 3Axe^{-3x}$$

$$y'' = -3Ae^{-3x} - 3Ae^{-3x} + 9Axe^{-3x}$$

$$= -6Ae^{-3x} + 9Axe^{-3x}$$

$$\Rightarrow \text{LHS of } (3) = y'' + 4y' + 3y$$

$$= -6Ae^{-3x} + \cancel{9Axe^{-3x}} + 4Ae^{-3x} \cancel{- 12Axe^{-3x}}$$

$$\cancel{+ 3Axe^{-3x}}$$

$$= -2Ae^{-3x}$$

RHS of (3) $= 5e^{-3x}$

$$\Rightarrow \quad -2A = 5 \quad \Rightarrow A = -\frac{5}{2}.$$

Hence $y_p = -\frac{5}{2}xe^{-3x}$ is a particular soln.

E.g 3:
$$y'' + 6y' + 9y = 5e^{-3x}.$$

In this example, the test function

$$y = Ae^{-3x} \quad \text{and}$$

$$y = Axe^{-3x}$$

do not work.

The right test function is

$$y = Ax^2e^{-3x} \; !$$

Ex: use the above test function to find a particular soln. (that is, determine the value of A)

Thm: Consider the D.E

$$ay'' + by' + cy = C_0 x^m e^{rx} \qquad (4)$$

where $m \geq 0$ is an integer, r is a real number $C_0 \in \mathbb{R}$

(If $r = 0$, then RHS of $(4) = C_0 x^m$)

Its associated characteristic/auxiliary eqn is

$$a\lambda^2 + b\lambda + c = 0 \qquad (5)$$

(I) If r is not a root of (5), then use the test function

$$y_p = (A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{rx}$$

Here $A_m, A_{m-1}, \ldots, A_1, A_0$ are to be determined

(II). If r is a simple ($\Leftrightarrow$ not repeated) root of (5), then use the test function

$$y_p = x(A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{rx}$$

(III) If r is a repeated root of (5), then use the test function

$$y_p = x^2(A_m x^m + A_{m-1}x^{m-1} + \cdots + A_1 x + A_0)e^{rx}$$

Recall the examples we did

E.g 1: $y'' - 4y' + 4y = 5e^{-3x} = \underbrace{5x^0 e^{-3x}}$

$$\Downarrow$$

$$\begin{cases} C_0 = 5 \\ m = 0 \\ r = -3 \end{cases}$$

We used test function

$$y = Ae^{-3x}$$

This is because the char. eqn

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 2$$

$\Rightarrow$ $r = -3$ is NOT a root of the char. eqn.

$\Rightarrow$ (I) use $y_p = A_0 e^{-3x}$

E.g 2: $y'' + 4y' + 3y = 5e^{-3x}$

$$\Rightarrow \begin{cases} C_0 = 5 \\ m = 0 \\ r = -3 \end{cases}$$

We used the test function

$$y = Axe^{-3x}$$

This is because the char. eqn is

$$\lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+1)(\lambda+3) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -3 \qquad \lambda_1 \neq \lambda_2$$

$\Rightarrow$ $r = -3$ is a root and is a simple root

$\Rightarrow$ (II)

$$y_p = xA_0e^{-3x} = A_0xe^{-3x}$$

E.g 3: $y'' + 6y' + 9y = 5e^{-3x}$

we used the test function

$$y = Ax^2 e^{-3x}$$

This is because the char. eqn

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda+3)^2 = 0$$

$$\lambda_1 = \lambda_2 = -3$$

$\Rightarrow$ $r = -3$ is a repeated root of the char. eqn

$\Rightarrow$ (III), use

$$y_p = x^2 A_0 e^{-3x}$$

Thm: Consider the D.E

$$ay'' + by' + cy = \begin{cases} C_0 x^m e^{\alpha x} \cos(\beta x) \\ \text{or} \\ C_0 x^m e^{\alpha x} \sin(\beta x) \end{cases}$$

Here $\begin{cases} a, b, c \in \mathbb{R} \\ m \geq 0 \text{ integer} \\ \alpha, \beta \in \mathbb{R} \end{cases}$ (Here $\beta \neq 0$)

Its char. eqn is

$$a\lambda^2 + b\lambda + c = 0 \qquad (6)$$

(I) If $\alpha \pm i\beta$ is not a root of (6),

then try test function

$$y = (A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{\alpha x} \cos(\beta x)$$
$$+ (B_m x^m + B_{m-1} x^{m-1} + \cdots + B_1 x + B_0) e^{\alpha x} \sin(\beta x)$$

(II) If $\alpha \pm i\beta$ is a root of (6), then

try the test function

$$y = x(A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{\alpha x} \cos(\beta x)$$
$$+ x(B_m x^m + B_{m-1} x^{m-1} + \cdots + B_1 x + B_0) e^{\alpha x} \sin(\beta x)$$

E.g. Find a particular soln of

$$y'' + 4y = \sin(2x) = 1\cdot x^0 \cdot e^{0\cdot x}\sin(2x)$$

A: Step 1: write down the char. eqn

$$r^2 + 4 = 0 \Rightarrow r^2 = -4 \quad (7)$$

$$\Rightarrow \quad r = \pm 2i$$

Since on the RHS of D.E

$$\sin(2x) = x^0 e^{0\cdot x} \sin(2x)$$

$$\Rightarrow \alpha = 0, \quad \beta = 2, \quad m = 0, \quad C_0 = 1$$

Note $\alpha \pm \beta i = \pm 2i$ is a root of (7).

Hence we should try (II)

$$y = B_0 x \sin(2x) + A_0 x \cos(2x) \quad (8)$$

Step 2: plug (8) into the D.E

$$\Rightarrow y' = B_0 \sin(2x) + 2B_0 x \cos(2x) + A_0 \cos(2x) - 2A_0 x \sin(2x)$$

$$y'' = 2B_0 \cos(2x) + 2B_0 \cos(2x) - 4B_0 x \sin(2x)$$
$$-2A_0 \sin(2x) - 2A_0 \sin(2x)$$
$$-4A_0 x \cos(2x)$$

$$= 4B_0 \cos(2x) - 4A_0 \sin(2x)$$
$$-4B_0 x \sin(2x) - 4A_0 x \cos(2x)$$

$$\Rightarrow \text{LHS} = y'' + 4y$$
$$= 4B_0 \cos(2x) - 4A_0 \sin(2x)$$

Note $\text{RHS} = \sin(2x)$

Compare:

sin-term: $-4A_0 = 1$ ①

cos-term: $4B_0 = 0$ ②

$$\Rightarrow \begin{cases} B_0 = 0 \\ A_0 = -\frac{1}{4} \end{cases}$$

Hence a particular soln:

$$y_p = -\frac{1}{4} x\cos(2x)$$

E.g: Find a particular soln.

$$y'' + 4y = \underbrace{t + \sin(2t)}_{\text{"split them"}}$$

A: Step 1: Solve $y'' + 4y = t$ (9)

Since RHS $= t = t^1 e^{0\cdot t}$

$\Rightarrow m = 1, \quad r = 0$

Note $r = 0$ is not a root of the char. eqn $\lambda^2 + 4 = 0$. $\Rightarrow$ Case (I)

We should try

$$y = (A_1 t + A_0) e^{0 \cdot x} = A_1 t + A_0$$

plugging into (9) $\Rightarrow$

$$LHS = \underbrace{0}_{y''} + \underbrace{4}_{4}\underbrace{(A_1 t + A_0)}_{y} = t = RHS$$

$$\Rightarrow \quad 4A_1 t + 4A_0 = t$$

Compare:

t-term: $4A_1 = 1$

Constant-term: $4A_0 = 0$

$$\Rightarrow \begin{cases} A_0 = 0 \\ A_1 = \frac{1}{4} \end{cases}$$

Hence a particular soln of (9)

$$y_{p,1} = \frac{1}{4} t$$

Step 2: Find a particular soln of

$$y'' + y = \sin(2t)$$

Already did in the previous e.g.

A particular soln:

$$y_{p,2} = -\frac{1}{4} t \cos(2t)$$

Step 3: Add the two particular solns:

$$y_p = y_{p,1} + y_{p,2}$$

$$= \frac{1}{4} t - \frac{1}{4} t \cos(2t)$$

this is a particular soln of the original D.E:

$$y'' + 4y = t + \sin(2t)$$