

The exam starts here: The exam has 42 points in total, but we will grade it out of 40 points. That means we will add up all the points that you get, but the maximum total points you can have is 40.

- (1) 1. (a) Make sure you have read the exam instruction and the integrity pledge. Then copy the line “I excel with integrity” and then **sign and date**. You don’t need to do this on a separated paper. It can be on the same papers where you are going to do the remaining questions.
- (9) 1. (b) Solve the following initial value problem. Show your work. You can leave final answer **in implicit form**;

$$\frac{dy}{dx} = \frac{x^4 e^y}{(1+x^5)^2}, \quad y(0) = 0.$$

Solution:

By writing our differential equation as

$$\frac{dy}{dx} = \frac{x^4}{(1+x^5)^2} \cdot e^y$$

we see that it is separable. So dividing both sides by e^y and integrating gives

$$\int e^{-y} dy = \int \frac{x^4}{(1+x^5)^2} dx.$$

The left hand side integral is easy and works out to

$$\int e^{-y} dy = -e^{-y} + C.$$

For our right side integral, we use the u -substitution $u = 1 + x^5$, which gives us that $du = 5x^4$. Then our integral works out to

$$\int \frac{x^4}{(1+x^5)^2} dx = \frac{1}{5} \int \frac{1}{u^2} = -\frac{1}{5u} = -\frac{1}{5(1+x^5)} + C.$$

So we get that our general solution is

$$e^{-y} = \frac{1}{5(1+x^5)} + C$$

after multiplying both of our integrals by -1 . To determine C , we plug in our point $(0, 0)$ from our initial condition:

$$1 = e^{-0} = \frac{1}{5(1+0^5)} + C = \frac{1}{5} + C.$$

From this we see that $C = 4/5$, so our final solution is

$$e^{-y} = \frac{1}{5(1+x^5)} + \frac{4}{5}.$$

(9) 2. Find the explicit solution of the following initial value problem. Show your work.

$$xy' + 2y = \sin x, \quad y\left(\frac{\pi}{2}\right) = 0.$$

Solution:

Our equation is first order and linear, so we turn it into the standard form by dividing both sides by x :

$$y' + \frac{2}{x}y = \frac{\sin x}{x}.$$

We then determine our integration factor:

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2.$$

Thus multiplying both sides of our equation by μ gives us

$$x^2 y' + 2xy = x \sin x$$

which can be rewritten

$$\frac{d}{dx}(x^2 y) = x \sin x.$$

Integrating both sides with respect to x gives us

$$x^2 y = \int x \sin x dx.$$

To integrate the left hand side, we use integration by parts with $u = x$ and $dv = \sin x dx$. This gives us that

$$\int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C.$$

Plugging in this solution to our integral into our previous equation and dividing both sides by x^2 gives us an explicit general solution

$$y(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2} + \frac{C}{x^2}.$$

To determine C , we plug in the point $(\pi/2, 0)$ from our initial condition to get

$$0 = -\frac{0}{\pi/2} + \frac{1}{(\pi/2)^2} + \frac{C}{(\pi/2)^2}$$

which gives us that

$$C = -1.$$

Thus our final solution is

$$y(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2} - \frac{1}{x^2}.$$

(10) 3. Find the explicit solution of the following initial value problem. Show your work.

$$(x^2 \sin x + y)dx - xdy = 0, \quad y\left(\frac{\pi}{2}\right) = 1.$$

Solution:

Let

$$M(x, y) = x^2 \sin x + y \text{ and } N(x, y) = -x.$$

We first get our partial derivatives

$$\frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = -1.$$

Since these partials are not equal the equation is not exact. However, we can notice that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2}{-x}$$

is a function of only x . So we can get an integration factor

$$\mu(x) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx} = e^{\int -\frac{2}{x} dx} = e^{-2 \ln x} = x^{-2}.$$

Multiplying our original equation by μ gives us the exact equation

$$\left(\sin x + \frac{y}{x^2}\right) dx - \frac{1}{x} dy = 0.$$

Let

$$M_0(x, y) = \sin x + \frac{y}{x^2} \text{ and } N_0(x, y) = -\frac{1}{x}.$$

Since our equation is exact, we know that there exists an function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M_0 \text{ and } \frac{\partial F}{\partial y} = N_0.$$

Using the second equality, tells us that

$$F(x, y) = \int N_0(x, y) dy = \int -\frac{1}{x} dy = -\frac{y}{x} + f(x).$$

Using the first equality from before gives us that

$$\sin x + \frac{y}{x^2} = M_0(x, y) = \frac{\partial F}{\partial x} = \frac{y}{x^2} + f'(x).$$

This tells us that $f'(x) = \sin x$ so $f(x) = -\cos x$. Therefore our general solution is $F(x, y) = C$ which is just

$$-\frac{y}{x} - \cos x = C$$

which we can solve for y to get

$$y(x) = C'x - x \cos x$$

where $C' = -C$. Finally, to determine C' , we plug in our initial condition point $(\pi/2, 1)$ to get

$$1 = \frac{\pi C'}{2} - 0$$

so $C' = 2/\pi$. Thus our final solution is

$$y(x) = \frac{2x}{\pi} - x \cos x.$$

(10) 4 (a). Find the explicit solution of the following initial value problem. Show your work.

$$2y'' - 2y' + y = 0, \quad y(0) = 2, y'(0) = 2.$$

Solution:

We look at our auxiliary equation

$$2\lambda^2 - 2\lambda + 1 = 0.$$

Since this polynomial does not factor, we use the quadratic equation to get

$$\lambda = \frac{2 \pm \sqrt{4 - 8}}{4} = \frac{2 \pm 2i}{4} = \frac{1}{2} \pm \frac{1}{2}i.$$

We use standard formula for solutions of second order constant coefficient homogeneous equations to get our general solution

$$y(x) = c_1 e^{x/2} \cos\left(\frac{x}{2}\right) + c_2 e^{x/2} \sin\left(\frac{x}{2}\right).$$

To determine c_1 and c_2 , we first take the derivative of our general solution to get

$$y'(x) = \frac{c_1}{2} e^{x/2} \cos\left(\frac{x}{2}\right) - \frac{c_1}{2} e^{x/2} \sin\left(\frac{x}{2}\right) + \frac{c_2}{2} e^{x/2} \sin\left(\frac{x}{2}\right) + \frac{c_2}{2} e^{x/2} \cos\left(\frac{x}{2}\right).$$

Plugging in our two initial conditions, we get the system of linear equation

$$2 = y(0) = c_1 \tag{1}$$

$$2 = y'(0) = \frac{c_1}{2} + \frac{c_2}{2} \tag{2}$$

Equation (1) tells us that $c_1 = 2$ and equation (2) tells us that

$$c_2 = 4 - c_1 = 2.$$

Thus our final solution is

$$y(x) = 2e^{x/2} \cos\left(\frac{x}{2}\right) + 2e^{x/2} \sin\left(\frac{x}{2}\right).$$

(3) 4(b). Let a, b, c be three real numbers such that the following two statements both hold:

1. $ax^2ydx + (\frac{1}{3}x^3 + e^y \cos y)dy = 0$ is exact.
2. $y = e^{3x} + xe^{3x}$ is a solution to $ay'' + by' + cy = 0$.

Find the values of a, b, c . Show your work.

Solution:

If our first equation is exact, we know that

$$\frac{\partial}{\partial y}(ax^2y) = \frac{\partial}{\partial x}\left(\frac{1}{3}x^3 + e^y \cos y\right).$$

After working out the partial derivatives, we get that

$$ax^2 = x^2$$

so $a = 1$.

Now looking at our second equation, since $y(x) = e^{3x} + xe^{3x}$ is a solution, we know that the auxiliary equation $a\lambda^2 + b\lambda + c = 0$ has a single repeated root of $\lambda = 3$. Since our coefficient of λ^2 is a which is equal to 1, we know that

$$a\lambda^2 + b\lambda + c = \lambda^2 + b\lambda + c = (\lambda - 3)^2 = \lambda^2 - 6\lambda + 9.$$

Therefore $b = -6$ and $c = 9$.

The exam ends here.