The exam starts here: The exam has 41 points in total including the bonus question, but we will grade it out of 40 points. That means we will add up all the points that you get, but the maximum total points you can have is 40.

(1) **1. (a)** Make sure you have read the exam instruction and the integrity pledge. Then copy the line "I excel with integrity" and then **sign and date**. You don't need to do this on a separated paper. It can be on the same papers where you are going to do the remaining questions.

(8) **1. (b)** Solve the following initial value problem of the Cauchy-Euler equation. Show your work.

$$x^2y'' - xy' + y = 0, \quad y(1) = 1, \; y'(1) = 3.$$

Solution

We first look at our characteristic equation

$$r^2 + (-1-1)r + 1 = r^2 - 2r + 1 = (r-1)^2 = 0$$

which has a double root at $r = 1$. Therefore our general solution is

$$y(t) = c_1 t + c_2 t \ln t$$

which has first derivative

$$y'(t) = c_1 + c_2 + c_2 \ln t.$$

Plugging in our initial conditions, we get that

$$\begin{aligned} 1 &= y(1) = c_1 \\ 3 &= y'(1) = c_1 + c_2. \end{aligned}$$

Therefore $c_1 = 1$ and $c_2 = 2$ and our solution is

$$y(t) = t + 2t \ln t.$$

(9) **2.** Use the method of **undetermined coefficients** to solve the following nonhomogeneous equation. **Find its general solution**. Show your work.

$$y'' - y = e^{2x}.$$

Solution

Since our $f(x) = e^{2x}$, we should have a particular solution of the form $y_p(x) = Ae^{2x}$. Notice that the second derivative is $y_p''(x) = 4Ae^{2x}$. Therefore

$$e^{2x} = 4Ae^{2x} - Ae^{2x} = 3Ae^{2x}$$

so $A = 1/3$.

The next step is to find the homogeneous solution. Our characteristic equation is

$$\lambda^2 - 1 = 0$$

which has two roots, namely 1 and -1. Therefore our homogeneous solution is

$$y_g(x) = c_1e^x + c_2e^{-x}.$$

Thus our general solution is

$$y(x) = c_1e^x + c_2e^{-x} + \frac{1}{3}e^{2x}.$$

(9) **3.** Use **variation of parameters** to solve the following nonhomogeneous equation:

$$y'' + y = \frac{1}{\sin x}.$$

Find its general solution. Show your work.

Solution

We first need to find two linearly independent solutions to the homogeneous system. Our characteristic equation is

$$\lambda^2 + 1 = 0$$

which has two roots, namely i and $-i$. Therefore, our two linearly independent solutions are $y_1(x) = \cos x$ and $y_2(x) = \sin x$. We now calculate the Wronskian

$$W[y_1, y_2](x) = \cos x \cdot \cos x + \sin x \cdot (-\sin x) = \cos^2 x + \sin^2 x = 1.$$

From variation of parameters, we know that there exists a particular solution of the form

$$y_p(x) = y_1(x)v_1(x) + y_2(x)v_2(x).$$

We first calculate v_1:

$$v_1(x) = \int \frac{-f(x)y_2(x)}{aW[y_1, y_2](x)} dx = -\int \frac{\sin x}{\sin x} dx = -x.$$

We can ignore the constant of integration here since it won't matter in the end. Now we calculate v_2:

$$v_2(x) = \int \frac{f(x)y_1(x)}{aW[y_1, y_2](x)} dx = \int \frac{\cos x}{\sin x} dx = \int \cot x \; dx = \ln|\sin x|.$$

Again, we can ignore the constant of integration here. Plugging it all in, we get that our particular solution is

$$y_p(x) = -x\cos x + \sin x(\ln|\sin x|).$$

Hence the general solution is $y = y_p + c_1 y_1 + c_2 y_2 = -x\cos x + \sin x(\ln|\sin x|) + c_1 \cos x + c_2 \sin x$.

(6) **4 (a).** Assume $y(t)$ satisfies the following differential equation and initial conditions:

$$y'' + y' = u(t-1) + 2, \qquad y(0) = 0,\ y'(0) = 1.$$

Find the Laplace transform $Y = \mathscr{L}\{y(t)\}$. Here u denotes the step function as in lectures. Show your work. (**Only need to find $Y = \mathscr{L}\{y(t)\}$. No need to solve for** $y(t)$.) You can leave your final answer unsimplified.

Solution

First we calculate the Laplace transform of the left hand side:

$\mathscr{L}\left\{y'' + y'\right\}(s) = \mathscr{L}\left\{y''\right\}(s) + \mathscr{L}\left\{y'\right\}(s) = s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) = (s^2+s)Y(s) - 1.$

Now we calculate the Laplace transform of the right hand side:

$$\mathscr{L}\left\{u(t-1) + 2\right\} = \mathscr{L}\left\{u(t-1)\right\} + 2\mathscr{L}\left\{1\right\} = \frac{e^{-s}}{s} + \frac{2}{s} = \frac{e^{-s}+2}{s}.$$

Now it's just algebra to solve for $Y(s)$:

$$\begin{aligned}(s^2+s)Y(s) - 1 &= \frac{e^{-s}+2}{s}\\ (s^2+s)Y(s) &= \frac{e^{-s}+2+s}{s}\\ Y(s) &= \frac{e^{-s}+2+s}{s(s^2+s)}\end{aligned}$$

(7) **4 (b).** Find the inverse Laplace transforms $\mathscr{L}^{-1}\{\frac{1}{s(s-1)}\}$ and $\mathscr{L}^{-1}\{\frac{e^{-s}}{s(s-1)}\}$. Show your work.

Solution

For the first Laplace transform, we first use partial fraction decomposition

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}.$$

This tells us that

$$As - A + Bs = 1$$

so we have the system of linear equations

$$\begin{aligned}A + B &= 0\\ -A &= 1.\end{aligned}$$

Therefore $A = -1$ and $B = 1$ so we get that

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \mathscr{L}^{-1}\left\{-\frac{1}{s} + \frac{1}{s-1}\right\} = -\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} = -1 + e^t.$$

Finally, for the second inverse Laplace transforms, we use the shifting property to get that

$$\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s(s-1)}\right\}(t) = u(t-1)\mathscr{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}(t-1) = u(t-1)(e^{t-1} - 1).$$

Bonus Question. The question 5 is worth only 1 point.

(2) **5.** Do you have any suggestions to improve the quality of future classes? Whatever suggestions you make, you will get the 1 point for this problem. You can put "No" if you do have nothing to say; and you still get the 1 point. Your suggestions are highly appreciated.

Thank you very much for all your suggestions.

The exam ends here.