Spring 2021 Math 20D Lecture B Homework #1

Topics covered: section 1.2, 2.2-2.4

1. Verify that the given relation is an implicit solution to the given differential equation.

$$y - \ln y = x^2 + 1, \quad \frac{dy}{dx} = \frac{2xy}{y-1}.$$

Answer. $y' - \frac{y'}{y} = 2x$, thus $y' = 2x/(1-1/y) = 2xy/(y-1)$.

2. Verify that the function $\phi(x) = c_1 e^x + c_2 e^{-2x}$ is a solution to the given equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that the following initial conditions is satisfied:

$$y(1) = 1, \quad y'(1) = 0.$$

Answer. Verification is omitted. $c_1 = \frac{2}{3e}$ and $c_2 = \frac{e^2}{3}$.

3. Solve the initial value problem

$$\sqrt{y}dx + (1+x)dy = 0, \quad y(0) = 1.$$

Answer. Separating variables we get $\frac{1}{\sqrt{y}}dy = -\frac{1}{1+x}dx$, integrating both sides we get $2y^{\frac{1}{2}} = -\ln|1+x| + C$. Plug in $y(0) = 1$, thus $C = 2$. The solution is

$$y = (\frac{-\ln|1+x|+2}{2})^2.$$

4. Solve the initial value problem

$$y' = x^3(1+y), \quad y(0) = 3.$$

Answer. Separating variables we get $\frac{1}{1+y}dy = x^3dx$, integrating both sides we get $\ln|1+y| = \frac{x^4}{4} + C$. Thus $|1+y| = e^{\frac{x^4}{4}+C}$ and $1 + y = \pm e^{\frac{x^4}{4}} \cdot e^C$. Let $C_1 = \pm e^C$, the general solution can be written as $y = -1 + C_1 e^{\frac{x^4}{4}}$. Plug in initial condition $y(0) = 3$, we get $C_1 = 4$, thus the solution to the IVP is $y = -1 + 4e^{\frac{x^4}{4}}$.

5. According to Newton's law of cooling, if an object at temperature T is immersed in a medium having the constant temperature M, then the rate of change of T is proportional to the difference of temperature $M - T$. This gives the differential equation

$$\frac{dT}{dt} = k(M - T).$$

(a) Solve the differential equation for T.

(b) A thermometer reading 100°F is placed in a medium having a constant temperature of 70°F. After 6 min, the thermometer reads 80°F. What is the reading after 20 min?

Answer.

(a) Separating variables $\frac{1}{M-T}dT = kdt$, integrating we get $-\ln|M-T| = kt + C$, thus $|M-T| = e^{-kt-C}$ and $M - T = \pm e^{-kt-C}$. Let $C_1 = \pm e^{-C}$, the general solution can be written as $T = M - C_1 e^{-kt}$.

(b) We know that $M = 70$. From $T(0) = 100$ we get $C_1 = -30$. From $T(6) = 80$ we get $k = \frac{\ln 3}{6}$. Thus $T(t) = 70 + 30e^{-\frac{\ln 3}{6}t}$, thus $T(20) = 70 + 30e^{-\frac{10\ln 3}{3}}$.

6. Solve the initial value problem

$$\frac{dy}{dx} + \frac{3y}{x} + 2 = 3x, \quad y(1) = 1.$$

Answer. The linear equation in standard form is $y' + \frac{3}{x}y = 3x - 2$, thus $\mu = e^{\int \frac{3}{x}dx} = x^3$. Multiply μ to the equation, we get $x^3y' + 3x^2y = x^3(3x-2)$. Thus $D_x[x^3y] = x^3(3x-2)$. Integrating we get $x^3y = \int x^3(3x-2)dx = \frac{3x^5}{5} - \frac{x^4}{2} + C$. Thus the general solution is $y = \frac{\frac{3x^5}{5} - \frac{x^4}{2} + C}{x^3} = \frac{3x^2}{5} - \frac{x}{2} + \frac{C}{x^3}$. Plug in the initial condition $y(1) = 1$, we get $C = \frac{9}{10}$.

7. Solve the initial value problem

$$(\sin x)\frac{dy}{dx} + y\cos x = x\sin x, \quad y(\pi/2) = 2.$$

Answer. Dividing by $\sin x$ and restricting the domain to $x \in [\pi/2, \pi)$ so that $\sin x \neq 0$, the linear equation in standard form is $y' + (\cot x)y = x$, thus $\mu = e^{\int(\cot x)dx} = e^{\ln(\sin x)} = \sin x$. Multiply μ on both sides of the equation, we get $(\sin x)y' + y\cos x = x\sin x$. (Note: we do not need the absolute value signs around $\sin x$ in μ because $\sin x > 0$ over our specified domain). Thus $D_x[(\sin x)y] = x\sin x$, integrating both sides, we get $(\sin x)y = -x\cos x + \sin x + C$. Thus $y = \frac{-x\cos x + \sin x + C}{\sin x}$. Plug in initial condition, we get $C = 1$.

Alternate solution. Notice that $[(\sin x)\frac{dy}{dx} + y\cos x] = D_x[(\sin x)y]$, so we can integrate both sides and get $(\sin x)y = -x\cos x + \sin x + C$. Dividing by $\sin x$ and restricting the domain to $x \in [\pi/2, \pi)$ so that $\sin x \neq 0$, we get $y = \frac{-x\cos x + \sin x + C}{\sin x}$. Plug in initial condition, we get $C = 1$.

8. Solve the initial value problem

$$(ye^{xy} - 1/y)dx + (xe^{xy} + x/y^2)dy = 0, \quad y(1) = 1.$$

Answer. Let $M = ye^{xy} - 1/y$ and $N = xe^{xy} + x/y^2$. We have $M_y = e^{xy} + xye^{xy} + \frac{1}{y^2}$ and $N_y = e^{xy} + xye^{xy} + \frac{1}{y^2}$, thus $M_y = N_x$, it is an exact equation. $F(x,y) = \int M dx = e^{xy} - \frac{x}{y} + g(y)$. Moreover, $F_y = xe^{xy} + \frac{x}{y^2} + g'(y) = N = xe^{xy} + x/y^2$. Thus $g'(y) = 0$ and we can choose $g(y) = 0$. The general solution is $F(x,y) = e^{xy} - \frac{x}{y} = C$. Plug in initial condition, we get $C = F(1,1) = e - 1$.

9. Consider the differential equation and initial condition

$$\alpha tx + (3t^2 + 2\cos(x)\sin(x))x' = 0, \quad x(1) = \pi/4.$$

(a) Find a value of the parameter α such that the equation is exact.

(b) For this value of α, find the solution of the initial value problem in implicit form.

Answer.

(a) The equation is
$$\alpha tx dt + (3t^2 + 2\cos(x)\sin(x))dx = 0.$$
By calculating M_x and N_t, we get:
$$M_x = \alpha t, \quad N_t = 6t$$
For the equation to be exact, we need $M_x = N_t$, that is, $\alpha = 6$.

(b) Since the equation with $\alpha = 6$ is exact, we can find a function $F(t,x)$ so that $F_t = M, F_x = N$. i.e.
$$F_t = 6tx$$
$$F_x = 3t^2 + 2\cos(x)\sin(x)$$
Integrating the first equation, we obtain
$$F(t,x) = 3t^2 x + h(x)$$
Setting $F_x = N$ gives
$$F_x = 3t^2 + h'(x) = 3t^2 + 2\cos(x)\sin(x)$$
Thus $h'(x) = 2\cos(x)\sin(x)$, and $h(x) = \sin^2(x) + C$. It is enough to choose any one of the $h(x)$, and we choose $h(x) = \sin^2(x)$. Therefore the general solution of the exact equation is given implicitly by $F(t,x) = 3t^2 x + \sin^2(x) = c$. To determine the constant c, we substitute the initial condition $x(1) = \pi/4$. So the solution to the IVP is $3t^2 x + \sin^2(x) = 1/2 + 3\pi/4$.