MATH 20D — HOMEWORK 2 SOLUTIONS

Problem 1. Solve $(y - x^2 \cos x)dx - xdy = 0$.

Solution. We can rewrite this equation as

$$y - x^2 \cos x - x\frac{dy}{dx} = 0$$

which we notice is a first order linear differential equation. Thus we can rearrange it to our standard form

$$\frac{dy}{dx} - = \frac{y}{x} = -x \cos x.$$

We calculate our integration factor as

$$\mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}.$$

Multiplying our equation by μ gives us

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = -\cos x.$$

Rewriting the left hand side as a single derivative gives

$$\frac{d}{dx}\left(\frac{y}{x}\right) = -\cos x$$

and integrating both sides with respect to x gives us

$$\frac{y}{x} = \int -\cos x dx = -\sin x + C.$$

Thus our final solution is

$$y(x) = -x \sin x + Cx.$$

Problem 2. Solve $(2xy)dx + (y^2 - 3x^2)dy = 0$.

Solution. We have our two function

$$M(x, y) = 2xy$$
$$N(x, y) = y^2 - 3x^2.$$

Calculating the partial derivatives gives that

$$\frac{\partial M}{\partial y} = 2x$$
$$\frac{\partial N}{\partial x} = -6x$$

which shows that our equation is not exact. However, we have that

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-8x}{2xy} = \frac{-4}{y}$$

is a function of only y. Thus we can get an integrating factor

$$\mu(y) = e^{\int \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \right) dy} = e^{\int -\frac{4}{y} dy} = e^{-4 \ln y} = \frac{1}{y^4}.$$

Multiplying our original differential equation by $\mu(y)$ gives us the exact equation

$$\frac{2x}{y^3} dx + \left(\frac{1}{y^2} - \frac{3x^2}{y^4} \right) dy = 0.$$

Since the equation is exact, we have a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = \frac{2x}{y^3} \tag{1}$$

$$\frac{\partial F}{\partial y} = \frac{1}{y^2} - \frac{3x^2}{y^4}. \tag{2}$$

Thus using equation (1) we have

$$F(x, y) = \int 2xy^3 dx = \frac{x^2}{y^3} + g(y)$$

for some function $g(y)$. Using equation (2) we get that

$$\frac{-3x^2}{y^3} + g'(y) = \frac{1}{y^2} - \frac{3x^2}{y^4}.$$

This tells us that $g'(y) = 1/y^2$. Since we can take any antiderivative of $g'(y)$, we have that $g(y) = -1/y$. Thus our final solution to the differential equation is

$$\frac{x^2}{y^3} - \frac{1}{y} = C.$$

Problem 3. Solve the initial value problem

$$y'' + 4y' + 6y = 0, \; y(0) = 1, \; y'(0) = 0.$$

Solution. Since our equation is a second order linear differential equation, we look at the auxiliary equation

$$r^2 + 4r + 6 = 0.$$

Using the quadratic formula, we get that

$$r = \frac{4 \pm \sqrt{16 - 24}}{2} = -2 \pm i\sqrt{2}.$$

Thus the general solution to our differential equation is

$$y(t) = c_1 e^{-2t} \cos(t\sqrt{2}) + c_2 e^{-2t} \sin(t\sqrt{2}).$$

We calculate the first derivative

$$y'(t) = -2c_1 e^{-2t} \cos(t\sqrt{2}) - \sqrt{2} c_1 e^{-2t} \sin(t\sqrt{2}) - 2c_2 e^{-2t} \sin(t\sqrt{2}) + \sqrt{2} c_2 e^{-2t} \cos(t\sqrt{2}).$$

Plugging in our initial conditions gives us the system of linear equations

$$y(0) = 1 = c_1$$

$$y'(0) = 0 = -2c_1 + \sqrt{2} c_2.$$

Solving this system gives us that $c_1 = 1$ and $c_2 = \sqrt{2}$. Thus our final solution is

$$y(t) = e^{-2t}\cos(t\sqrt{2}) + \sqrt{2}e^{-2t}\sin(t\sqrt{2}).$$

Problem 4. Solve the initial value problem

$$9y'' + 6y' + y = 0,\ y(0) = 6,\ y'(0) = -1.$$

Solution. Since this differential equation is second order and linear, we look at the auxiliary equation

$$9r^2 + 6r + 1 = 0.$$

This equation can be factored as $(3r + 1)^2 = 0$ which has a single solution of $r = -1/3$. Thus our general solution is

$$y(t) = c_1 e^{-\frac{t}{3}} + c_2 t e^{-\frac{t}{3}}.$$

We take our first derivative

$$y'(t) = -\frac{c_1}{3}e^{-\frac{t}{3}} + c_2 e^{-\frac{t}{3}} - \frac{c_2}{3}te^{-\frac{t}{3}}.$$

Plugging in our initial values gives us the system of linear equations

$$y(0) = 6 = c_1$$
$$y'(0) = -1 = -\frac{c_1}{3} + c_2.$$

Solving this system gives us that $c_1 = 6$ and $c_2 = 1$. Thus our final solution is

$$y(t) = 6e^{-\frac{t}{3}} + te^{-\frac{t}{3}}.$$

Problem 5. Solve the initial value problem

$$y'' + y' - 2y = 0,\ y(0) = \alpha,\ y'(0) = 1.$$

Also, find the value of α so that the solution goes to 0 when $t \to +\infty$.

Solution. Since our differential equation is first order and linear, we look at the auxiliary equation

$$r^2 + r - 2 = 0.$$

This can be factored as $(r + 2)(r - 1) = 0$ so we have two solutions, $r = -2$ and $r = 1$. Therefore our general solution to our differential equation is

$$y(t) = c_1 e^t + c_2 e^{-2t}.$$

Taking the first derivative gives

$$y'(t) = c_1 e^t - 2c_2 e^{-2t}.$$

Plugging in our initial values gives us the system of linear equations

$$y(0) = \alpha = c_1 + c_2$$
$$y'(0) = 1 = c_1 - 2c_2.$$

Subtracting the second equation from the first gives

$$\alpha - 1 = 3c_2$$

so $c_2 = \frac{\alpha-1}{3}$. Then we get that

$$c_1 = \alpha - c_2 = \alpha - \frac{\alpha - 1}{3} = \frac{2\alpha + 1}{3}.$$

Thus our final solution is

$$y(t) = \frac{2\alpha + 1}{3} e^t + \frac{\alpha - 1}{3} e^{-2t}.$$

For the second part of the problem, for any constants c_1 and c_2, we have that

$$\lim_{t\to\infty} (c_1 e^t + c_2 e^{-2t} = \lim_{t\to\infty} (c_1 e^t) + c_2 \lim_{t\to\infty} e^{-2t} = \lim_{t\to\infty} (c_1 e^t)$$

since $\lim_{t\to\infty} e^{-2t} = 0$. Also, since $\lim_{t\to\infty} e^t = \infty$, we have that $\lim_{t\to\infty} y(t) = 0$ if and only if $c_1 = 0$. In our case, since $c_1 = \frac{2\alpha+1}{3}$, we have that $\lim_{t\to\infty} y(t) = 0$ if and only if

$$\frac{2\alpha + 1}{3} = 0$$

which happens if and only if

$$\alpha = -\frac{1}{2}.$$