Spring 2021 Math 20D Lecture B Homework #3

1. Find a particular solution to the differential equation

$$y'' + 2y' - y = 10 + x.$$

Solution: One makes the "guess" that y_p has the form $y_p = Ax + B$ for suitable constants A and B. Note that $y_p' = A,\ y_p'' = 0$, so that

$$y_p'' + 2y_p' - y_p = 0 + 2A - Ax - B = x - 10.$$

This yields the equations

$$-Ax = x,\ 2A - B = 10 \implies A = -1, B = -12.$$

Thus a particular solution is

$$y_p = -x - 12.$$

2. Find a particular solution to the differential equation

$$2z'' + z = 9e^{2t}.$$

Solution: The characteristic equation corresponding to the homogeneous equation $2z'' + z = 0$ is $2\lambda^2 + \lambda = 0$, which has solutions

$$\lambda_{1,2} = \frac{-0 \pm \sqrt{0^2 - 4 * 2}}{4} = \frac{\pm 2i\sqrt{2}}{4} = \pm\frac{i}{\sqrt{2}}.$$

These roots correspond to a homogeneous solution of

$$z_h = c_1 \cos\left(\frac{t}{\sqrt{2}}\right) + c_2 \sin\left(\frac{t}{\sqrt{2}}\right).$$

Since this solution doesn't involve e^{2t}, one makes the guess that $z_p = Ae^{2t}$ for an undetermined constant A. Noting that $z_p' = 2Ae^{2t}$ and $z_p'' = 4Ae^{2t}$, one arrives at

$$2z_p'' + z_p = 8Ae^{2t} + Ae^{2t} = 9e^{2t} \implies A = 1.$$

Thus a particular solution is $z_p = e^{2t}$.

3. Find a particular solution to the differential equation

$$4y'' + 11y' - 3y = -2xe^{-3x}.$$

Solution: The characteristic equation corresponding to the homogeneous equation $4y'' + 11y' - 3y = 0$ is $4\lambda^2 + 11\lambda - 3 = 0$, which has solutions

$$\lambda_{1,2} = \frac{-11 \pm \sqrt{(11)^2 - 4 * (-3) * 4}}{4 * 2} = -3,\ \frac{1}{4}.$$

Now, one guesses that a particular solution has the form $y_p = x^s(Ax+b)e^{-3x}$, and since -3 is a nonrepeated root of the auxiliary equation, it is the case that $s = 1$ so that $y_p = (Ax^2 + Bx)e^{-3x}$. Further,

$$y_p' = 2Axe^{-3x} - 3Ax^2e^{-3x} + Be^{-3x} - 3Bxe^{-3x}.$$

$$\begin{aligned} y_p'' &= 2Ae^{-3x} - 6Axe^{-3x} - 6Axe^{-3x} + 9Ax^2e^{-3x} - 3Be^{-3x} - 3Be^{-3x} + 9Bxe^{-3x} \\ &= e^{-3x}(-3Ax^2 + 2Ax - 3Bx + B) \\ &= e^{-3x}(9Ax^2 - 12Ax + 2A + 9Bx - 6B) \end{aligned}$$

Now, substituting this information into the original differential equation, one arrives at

$$4y_p'' + 11y_p' - 3y_p$$

$$\begin{aligned} &= e^{-3x}[4(9Ax^2 - 12Ax + 2A + 9Bx - 6B) \\ &\qquad + 11(-3Ax^2 + 2Ax - 3Bx + B) - 3(Ax^2 + Bx)] \\ &= e^{-3x}\left[-26Ax + 8A - 13B\right] = -2xe^{-3x} \\ &\implies 26A = 2 \implies A = \frac{1}{13} \\ &\implies \frac{8}{13} - 13B = 0 \implies B = \frac{8}{13^2} \end{aligned}$$

So a particular solution to the given differential equation is

$$y_p = \left(\frac{1}{13}x^2 + \frac{8}{169}x\right)e^{-3x}$$

4. Find a particular solution to the differential equation

$$y'' + 4y = 8\sin(2t)$$

Solution: The characteristic equation corresponding to the homogeneous problem has roots

$$\lambda_{1,2} = \frac{\pm\sqrt{-16}}{2} = \pm 2i$$

Note that these roots correspond to the general solution

$$y_g = c_1\cos(2t) + c_2\sin(2t).$$

As such, one guesses that a particular solution has the form $y_p = t^s(A\cos(2t) + B\sin(2t))$, and since $\sin(2t)$ appears in the general solution, $s = 1$ so that

$$y_p = At\cos(2t) + Bt\sin(2t).$$

Taking derivatives yields

$$y_p' = A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t)$$

$$y_p'' = -2A\sin(2t) - 2A\sin(2t) - 4At\cos(2t) + 2B\cos(2t) + 2B\cos(2t) - 4Bt\sin(2t)$$
$$= \sin(2t)(-4A - 4Bt) + \cos(2t)(-4At + 4B)$$

Plugging these back into the differential equation yields

$$y_p'' + 4y_p = \sin(2t)(-4A - 4Bt) + \cos(2t)(-4At + 4B) + 4[At\cos(2t) + Bt\sin(2t)]$$

$$= \sin(2t)(-4A - 4Bt + 4Bt) + \cos(2t)(-4At + 4B + 4At) = 8\sin(2t).$$

Grouping terms produces the system of equations

$$-4A = 8,\ 4B = 0 \implies A = -2, B = 0.$$

So a particular solution to this differential equation is

$$y_p = -2t\cos(2t).$$