Spring 2021 Math 20D Lecture B Homework #4

Due Sunday, 11:59pm, May 2nd

Submit this homework through Gradescope.

Topics covered: section 4.6, 4.7, 7.2

1. Find the general solution to the differential equation

$$y'' - 6y' + 9y = t^{-3}e^{3t}.$$

SOLUTION: We begin by noticing that this is a linear nonhomogeneous equation which has a corresponding homogeneous equation with constant coefficients. Next, we check the right hand side of the equation and find that we do not have a test function which can be used to find a particular solution to this equation. Therefore, after we solve for our homogeneous solutions, we will need to use the variation of parameters method to solve the nonhomogeneous problem.

First we will solve for the homogeneous solutions: $y_h'' - 6y_h' + 9y_h = 0$. The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$, which is a perfect square, so we have a repeated root of $\lambda = 3$. The solution to the homogeneous equation is therefore $y_h = c_1e^{3t} + c_2te^{3t}$

Next we must find a particular solution y_p to the nonhomogeneous equation. We notice that $t^{-3}e^{3t}$ is not of the forms which we can solve with undetermined coefficients and therefore will use variation of parameters: $y_p = v_1y_1 + v_2y_2$. We therefore solve $v_1 = -\int \frac{y_2(t^{-3}e^{3t})}{y_1y_2' - y_1'y_2}dt = -\int \frac{t^{-2}e^{6t}}{e^{3t}[e^{3t}+3te^{3t}]-3te^{6t}}dt = -\int \frac{t^{-2}e^{6t}}{e^{6t}}dt = -\int \frac{dt}{t^2} = \frac{1}{t}$ and $v_2 = \int \frac{y_1(t^{-3}e^{3t})}{y_1y_2' - y_1'y_2}dt = \int \frac{dt}{t^3} = -\frac{1}{2t^2}$. All together, $y_p = \frac{e^{3t}}{t} - \frac{te^{3t}}{2t^2} = \frac{e^{3t}}{2t}$ and so the general solution, by the superposition principle, is $y = y_h + y_p = c_1e^{3t} + c_2te^{3t} + \frac{e^{3t}}{2t}$.

2. Find the general solution to the differential equation for $x > 0$.

$$\frac{d^2y}{dx^2} + \frac{6}{x}\frac{dy}{dx} + \frac{4}{x^2}y = 0.$$

SOLUTION: We notice that this is a Cauchy-Euler equation with solutions of the form x^r. The characteristic equation is $(1)r^2 + (6-1)r + 4 = r^2 + 5r + 4 = 0$, which factors to $(r+4)(r+1) = 0$. Therefore the general solution is $y = c_1x^{-1} + c_2x^{-4}$.

3. Solve the initial value problem for the Cauchy-Euler equation.

$$x^2y'' - 3xy' + 5y = 0, \quad y(1) = -1, \quad y'(1) = 4.$$

SOLUTION: The characteristic equation for this Cauchy-Euler equation is $(1)r^2 + (-3-1)r + 5 = r^2 - 4r + 5 = 0$ which has roots $r = 2 \pm i$ and so the general solution is $y = x^2[c_1\cos(\ln x) + c_2\sin(\ln x)]$ for $x > 0$. To find the particular solution we will take the derivative and solve the system of equations for c_1 and c_2. $y' = x[(2c_1 + c_2)\cos(\ln x) + (-c_1 + 2c_2)\sin(\ln x)]$, and substituting in the initial conditions we get $-1 = c_1, 4 = 2c_1 + c_2$, meaning $c_1 = -1, c_2 = 6$. Therefore our particular solution is: $y = x^2[-\cos(\ln x) + 6\sin(\ln x)]$.

4. Devise a modification of the method for Cauchy-Euler equation to find two linearly independent solutions to the given equation.

$$(x+1)^2 y'' + 10(x+1)y' + 14y = 0, \quad x > -1.$$

(Hint: Try $y_p = (x+1)^r$).

SOLUTION: We notice that this is a Cauchy-Euler equation with solutions of the form $(x+1)^r$ since $\frac{d}{dx}(x+1)^r = r(x+1)^{r-1}$ and $\frac{d^2}{d^2x}(x+1)^r = r(r-1)(x+1)^{r-2}$ (you can prove that this is a Cauchy-Euler equation using the change of variables $u = x+1$). The characteristic equation is $(1)r^2 + (10-1)r + 14 = r^2 + 9r + 14 = 0$, which factors to $(r+7)(r+2) = 0$. Therefore the general solution is $y = c_1 x^{-2} + c_2 x^{-7}$.

5. Find the general solution to the non-homogeneous Cauchy-Euler equation using variation of parameters.

$$x^2 y'' + 3xy' + y = \frac{1}{x}.$$

SOLUTION: We notice that this is a linear equation and therefore, by the superposition principle, the general solution will be of the form $y = y_h + y_p$, where y_h is the general solution to the corresponding homogeneous equation $x^2 y_h'' + 3xy_h' + y_h = 0$ and y_p is a particular solution to the nonhomogeneous equation which we can find using variation of parameters. y_h is the solution to a Cauchy-Euler equation with solutions of the form x^r and characteristic equation $(1)r^2 + (3-1)r + 1 = r^2 + 2r + 1 = (r+1)^2 = 0$. We have a repeated root $r = -1$ and therefore $y_h = c_1 x^{-1} + c_2 x^{-1}\ln x$. We now solve for $y_p = u_1 y_1 + u_2 y_2$. $u_1 = -\int \frac{y_2(\frac{1}{x^3})}{y_1 y_2' - y_1' y_2} dx = -\int \frac{(x^{-1}\ln x)(\frac{1}{x^3})}{x^{-1}[-x^{-2}\ln x + x^{-2}] - (-x^{-2})(x^{-1}\ln x)} dx = -\int \frac{x^{-4}\ln x}{x^{-3}} dx = -\int \frac{\ln x}{x} dx$ and by using the substitution $u = \ln x$, we get $u_1 = -\frac{(\ln x)^2}{2}$. $u_2 = \int \frac{y_1(\frac{1}{x^3})}{y_1 y_2' - y_1' y_2} dx = \int \frac{x^{-4}}{x^{-3}} dx = \int \frac{dx}{x} = \ln x$. Therefore $y_p = -\frac{(\ln x)^2}{2}(x^{-1}) + (\ln x)(x^{-1}\ln x) = \frac{x^{-1}(\ln x)^2}{2}$. The general solution is $y = y_h + y_p = c_1 x^{-1} + c_2 x^{-1}\ln x + \frac{x^{-1}(\ln x)^2}{2} = \frac{1}{x}[c_1 + c_2 \ln x + \frac{1}{2}(\ln x)^2]$.

6. Let

$$f(t) = \begin{cases} 1-t, & 0 \le t < 1, \\ 0, & t \ge 1. \end{cases}$$

Use the definition of Laplace transform to find $\mathscr{L}\{f(t)\}$.

SOLUTION: $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^1 e^{-st} f(t)dt + \int_1^\infty e^{-st} f(t)dt = \int_0^1 e^{-st}(1-t)dt + \int_1^\infty e^{-st}(0)dt = \int_0^1 e^{-st}(1-t)dt$. We then apply integration by parts with $u = 1-t$

(so $\frac{du}{dx} = -1$) and $\frac{dv}{dx} = e^{-st}$ (so $v = -\frac{e^{-st}}{s}$): $\int_0^1 e^{-st}(1-t)dt = [-\frac{e^{-st}}{s}(1-t)]_0^1 - \int_0^1(-1)(-\frac{e^{-st}}{s})dt = \frac{1}{s} - [-\frac{e^{-st}}{s^2}]_0^1 = \frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{1}{s^2} = \frac{e^{-s}+s-1}{s^2}$. This is discontinuous at $s = 0$. For $s = 0$, $\int_0^1 e^{-st}(1-t)dt = \int_0^1(1-t)dt = [t - \frac{t^2}{2}]_0^1 = \frac{1}{2}$. Therefore, $\mathscr{L}\{f(t)\} = \begin{cases} \frac{e^{-s}+s-1}{s^2}, & s \neq 0, \\ \frac{1}{2}, & s = 0. \end{cases}$

Note: Due to the number of students in this class, a subset of the questions will be used to grade this assignment. This subset will be determined after all of the assignments have been submitted.