Spring 2021 Math 20D Lecture B Homework #5

Topics covered: section 7.3, 7.4, 7.5

1. Determine $\mathscr{L}\{e^{7t}\cos^2 t\}$ and $\mathscr{L}\{te^{7t}\cos^2 t\}$. You can leave your answer unsimplified.
   Hint: $\cos^2 t = \frac{1}{2}\cos(2t) + \frac{1}{2}$.

   Answer. Note that $\cos^2 t = \frac{\cos(2t)+1}{2}$, thus

   $$\mathscr{L}\{e^{7t}\cos^2 t\} = \frac{1}{2}\mathscr{L}\{e^{7t}(\cos(2t)+1)\} = \frac{1}{2}[\frac{s-7}{(s-7)^2+4} + \frac{1}{s-7}].$$

   By Theorem 5 of §7.3, we have $\mathscr{L}\{te^{7t}\cos^2 t\} = (-1)\frac{d}{ds}(\mathscr{L}\{e^{7t}\cos^2 t\})$. This implies

   $$\mathscr{L}\{te^{7t}\cos^2 t\} = -\frac{1}{2}[\frac{4-(s-7)^2}{((s-7)^2+4)^2} - \frac{1}{(s-7)^2}].$$

2. Determine

   $$\mathscr{L}^{-1}\{\frac{3}{(2s+5)^3}\}.$$

   Answer. Note that $\frac{3}{(2s+5)^3} = \frac{3}{8(s+5/2)^3}$ and from the table we know $\mathscr{L}\{t^2\} = \frac{2}{s^3}$, thus $\mathscr{L}^{-1}\{\frac{3}{8(s+5/2)^3}\} = \frac{3}{16}t^2e^{-\frac{5}{2}t}$.

3. Determine

   $$\mathscr{L}^{-1}\{\frac{7s^2-41s+84}{(s-1)(s^2-4s+13)}\}.$$

   Answer. Let $F(s) = \frac{7s^2-41s+84}{(s-1)(s^2-4s+13)} = \frac{A}{s-1} + \frac{Bs+C}{(s-2)^2+9}$. Solve A, B, C, we get

   $$F(s) = \frac{5}{s-1} + \frac{2s-19}{(s-2)^2+9} = \frac{5}{s-1} + \frac{2(s-2)-15}{(s-2)^2+9} = \frac{5}{s-1} + \frac{2(s-2)}{(s-2)^2+9} - \frac{5\cdot 3}{(s-2)^2+9}.$$

   Thus $\mathscr{L}^{-1}\{F(s)\} = 5e^t + 2e^{2t}\cos(3t) - 5e^{2t}\sin(3t)$.

4. Solve the given initial value problem using the method of Laplace transforms.

   $$y'' + 6y' + 5y = 12e^t, \quad y(0) = -1,\ y'(0) = 7.$$

   Answer. Take Laplace transform $\mathscr{L}\{y'' + 6y' + 5y\} = \mathscr{L}\{12e^t\}$ we get

   $$s^2Y(s) - sy(0) - y'(0) + 6(sY(s) - y(0)) + 5Y(s) = \frac{12}{s-1},$$

   $$\Rightarrow s^2Y(s) + s - 7 + 6sY(s) + 6 + 5Y(s) = \frac{12}{s-1},$$

$$\Rightarrow (s^2+6s+5)Y(s) = -s+1+\frac{12}{s-1},$$
$$\Rightarrow (s^2+6s+5)Y(s) = \frac{-s^2+2s+11}{s-1},$$
$$\Rightarrow Y(s) = \frac{-s^2+2s+11}{(s-1)(s^2+6s+5)} = \frac{-s^2+2s+11}{(s-1)(s+1)(s+5)}.$$

Thus

$$Y(s) = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s+5},$$

and the solution to the initial value problem is $y(t) = \mathscr{L}^{-1}\{Y(s)\} = e^t - e^{-t} - e^{-5t}$.

5. Solve the given initial value problem using the method of Laplace transforms.

$$ty'' - 2y' + ty = 0; \quad y(0) = 1,\ y'(0) = 0.$$

Hint: $\mathscr{L}^{-1}\{1/(s^2+1)^2\} = (\sin t - t\cos t)/2$.

Answer. Take Laplace transform $\mathscr{L}\{ty'' - 2y' + ty\} = \mathscr{L}\{0\}$ we get

$$(-1)\frac{d}{ds}[s^2Y(s) - s] - 2(sY(s) - 1) + (-1)\frac{d}{ds}[Y(s)] = 0,$$
$$\Rightarrow -(s^2+1)Y'(s) - 4sY(s) + 3 = 0,$$
$$\Rightarrow Y'(s) + \frac{4s}{s^2+1}Y(s) = \frac{3}{s^2+1}.$$

The integrating factor μ for the linear equation is $\mu(s) = e^{\int \frac{4s}{s^2+1}ds} = (s^2+1)^2$. Thus $D_s[\mu(s)Y(s)] = 3(s^2+1)$ and

$$Y(s) = \frac{s^3+3s}{(s^2+1)^2} + \frac{C}{(s^2+1)^2}.$$

Note that we cannot get rid of the arbitrary constant C since we have $\lim_{s\to\infty} Y(s) = 0$ for any C. Using the hint, we get $\mathscr{L}\{\frac{C}{(s^2+1)^2}\} = C(\sin t - t\cos t)/2$. Let $F(s) = \frac{s^3+3s}{(s^2+1)^2} = \frac{s}{s^2+1} + \frac{2s}{(s^2+1)^2}$. Using the derivative property of Laplace transform, we observe that $\mathscr{L}\{t\sin t\} = (-1)\frac{d}{ds}[\frac{1}{s^2+1}] = \frac{2s}{(s^2+1)^2}$. Thus the solution to the initial value problem is

$$\mathscr{L}^{-1}\{Y(s)\} = \cos t + t\sin t + c(\sin t - t\cos t), \quad c \text{ arbitrary}.$$