Spring 2021 Math 20D Lecture B Homework #6

Due Sunday, 11:59pm, May 16th

Submit this homework through Gradescope.
Topics covered: section 7.6, 7.7/7.8

1. Express the given function using window and step functions and then compute its Laplace transform.

$$g(t) = \begin{cases} 0, & t \in [0,2), \\ t+1, & t \in [2,\infty). \end{cases}$$

SOLUTION: We can see that the function is zero for every value $t < 2$ and $t+1$ for every other value of t. Therefore, $g(t) = (t+1)u(t-2)$. We use the identity $\mathscr{L}\{f(t)u(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$ to take the Laplace transform of this function: $\mathscr{L}\{g(t)\} = e^{-2s}\mathscr{L}\{t+3\} = e^{-2s}[\frac{1}{s^2} + \frac{3}{s}]$.

2. Solve the given initial value problem using the method of Laplace transforms.

$$y'' + y = u(t-3), \quad y(0) = 0,\ y'(0) = 1.$$

SOLUTION: We take the Laplace transform of both sides, yielding $s^2Y(s) - 1 + Y(s) = \frac{e^{-3s}}{s}$, where $Y(s) = \mathscr{L}\{y(t)\}$. Solving for $Y(s)$, we get $Y(s) = e^{-3s}[\frac{1}{s(s^2+1)}] + \frac{1}{s^2+1}$. We will use the partial fraction decomposition: $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$. Taking the inverse Laplace transform,

$$y(t) = \mathscr{L}^{-1}\{\frac{1}{s^2+1}\} + \mathscr{L}^{-1}\{\frac{e^{-3s}}{s}\} - \mathscr{L}^{-1}\{e^{-3s}(\frac{s}{s^2+1})\} = \sin t + u(t-3)[1 - \cos(t-3)]$$

3. Use the convolution theorem to compute

$$\mathscr{L}^{-1}\{\frac{1}{(s^2+4)^2}\}.$$

SOLUTION: By convolution theorem,

$$\mathscr{L}^{-1}\{\frac{1}{(s^2+4)^2}\} = \frac{1}{4}\mathscr{L}^{-1}\{\frac{2}{s^2+4} \cdot \frac{2}{s^2+4}\} = \frac{1}{4}\sin(2t) * \sin(2t).$$

The convolution is

$$\begin{aligned} \sin(2t) * \sin(2t) &= \int_0^t \sin(2(t-\tau))\sin(2\tau)d\tau \\ &= \int_0^t \frac{1}{2}[\cos(4\tau - 2t) - \cos(2t)]d\tau \\ &= \frac{1}{2}[\frac{1}{4}\sin(4\tau - 2t) - \cos(2t)\tau]|_{\tau=0}^{t} \\ &= \frac{1}{2}[\frac{1}{2}\sin(2t) - t\cos(2t)] \\ &= \frac{1}{4}\sin(2t) - \frac{1}{2}t\cos(2t). \end{aligned}$$

The second equality uses the identity that $\sin(A)\sin(B) = \frac{1}{2}[\cos(B-A) - \cos(B+A)]$. Thus

$$\mathscr{L}^{-1}\{\frac{1}{(s^2+4)^2}\} = \frac{1}{16}\sin(2t) - \frac{1}{8}t\cos(2t).$$

4. Solve the integro-differential equation for $y(t)$:

$$y'(t) - 2\int_0^t e^{t-\tau}y(\tau)d\tau = t, \quad y(0) = 2.$$

SOLUTION: First, we recognize that $\int_0^t e^{t-\tau}y(\tau)d\tau = e^t * y(t)$. Taking Laplace transform of both sides, we get

$$sY(s) - y(0) - 2\mathscr{L}\{e^t\}Y(s) = \frac{1}{s^2},$$

$$\Rightarrow sY(s) - 2 - \frac{2Y(s)}{s-1} = \frac{1}{s^2},$$

$$\Rightarrow Y(s) = \frac{2s^3 - 2s^2 + s - 1}{s^2(s+1)(s-2)}.$$

Performing partial fraction decomposition, we get $Y(s) = (-\frac{3}{4})\frac{1}{s} + (\frac{1}{2})\frac{1}{s^2} + (\frac{3}{4})\frac{1}{s-2} + (2)\frac{1}{s+1}$ and therefore $y(t) = -\frac{3}{4} + \frac{t}{2} + \frac{3}{4}e^{2t} + 2e^{-t}$.