Spring 2021 Math 20D Lecture B Homework #7

Due Sunday, 11:59pm, May 23th

Submit this homework through Gradescope.
Topics covered: section 7.9, 8.2

1. Find the solution of the given initial value problem:

$$y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 1,\, y'(0) = 0.$$

Answer. Taking Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 2Y(s) = e^{-\pi s},$$

$$\Rightarrow Y(s) = \frac{s + 2 + e^{-\pi s}}{s^2 + 2s + 2} = \frac{s+1}{(s+1)^2 + 1} + \frac{1}{(s+1)^2 + 1} + \frac{e^{-\pi s}}{(s+1)^2 + 1}.$$

Let $F(s) = \frac{1}{(s+1)^2+1}$, then $f(t) = \mathscr{L}^{-1}\{F(s)\} = e^{-t}\sin(t)$, thus

$$\begin{aligned} y(t) &= e^{-t}\cos(t) + e^{-t}\sin(t) + f(t-\pi)u(t-\pi) \\ &= e^{-t}\cos(t) + e^{-t}\sin(t) + e^{-(t-\pi)}\sin(t-\pi)u(t-\pi). \end{aligned}$$

2. Find the solution of the given initial value problem:

$$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0,\, y'(0) = 0.$$

Answer. Taking Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = e^{-\pi s} - e^{-2\pi s},$$

$$\Rightarrow Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2 + 4}$$

Let $F(s) = \frac{1}{s^2+4}$, then $f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2}\sin(2t)$, thus

$$\begin{aligned} y(t) &= f(t-\pi)u(t-\pi) - f(t-2\pi)u(t-2\pi) \\ &= \frac{1}{2}\sin 2(t-\pi)u(t-\pi) - \frac{1}{2}\sin 2(t-2\pi)u(t-2\pi) \\ &= \frac{1}{2}\sin(2t)u(t-\pi) - \frac{1}{2}\sin(2t)u(t-2\pi). \end{aligned}$$

The last equality is using the fact that $\sin(x)$ *is* 2π *periodic.*

3. Determine the radius of convergence of the given power series:

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2}.$$

Answer. Write the power series in the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ we get

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x + \frac{1}{2})^n.$$

Thus $a_n = \frac{2^n}{n^2}$. Using the ratio test

$$\lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} \frac{2^n(n+1)^2}{n^2 2^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \frac{1}{2}.$$

So the radius of convergence is $\frac{1}{2}$.