Spring 2021 Math 20D Lecture B Homework #8

Due Sunday, 11:59pm, May 30th

Submit this homework through Gradescope.

Topics covered: section 8.3

1. Find a power series solution centered at $x = 0$ to the differential equation $y' + 2y = 0$. Your answer should include a general formula for the coefficients. Then explain why your solution is the same as $y = a_0 e^{-2x}$.

   Answer. Assume $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Plug into the equation

$$y' + 2y = \sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

   Shifting the summation index, we shall get

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + 2\sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\Rightarrow \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + 2\sum_{k=0}^{\infty} a_k x^k = 0,$$

   thus the recurrence relation is

$$(k+1)a_{k+1} + 2a_k = 0, \quad k = 0, 1, 2, \cdots$$

   and $a_{k+1} = \frac{-2a_k}{k+1}$. The general formula for a_k is

$$a_k = \frac{(-2)^k}{k!}a_0, \quad k = 0, 1, 2, \cdots.$$

   Therefore the power series solution for the differential equation is

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k.$$

   Note the Taylor series of e^t is $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$. Replacing t by $-2x$, we get $e^{-2x} = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k$. Hence the solution y is the same as $y = a_0 e^{-2x}$.

2. Solve the initial value problem using a power series centered at $x = 0$. Write out the first four nonzero terms of the infinite series:

$$y'' - xy' - y = 0, \quad y(0) = 2, \; y'(0) = -1.$$

Answer. Assume $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Plug into the equation

$$y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Shifting the summation index, we shall get

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0,$$

thus the recurrence relation is $2a_2 - a_0 = 0$ and

$$(k+2)(k+1)a_{k+2} - ka_k - a_k = 0, \quad k = 1, 2, \cdots,$$

thus $(k+2)a_{k+2} = a_k$. The first few terms are

$$a_3 = \frac{1}{3}a_1, \quad a_4 = \frac{1}{8}a_0, \quad a_5 = \frac{1}{15}a_1, \quad \cdots$$

Therefore

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots) + a_1(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots).$$

Since $y(0) = a_0$ and $y'(0) = a_1$, we get $a_0 = 2$ and $a_1 = -1$. The first four nonzero terms of the power series solution is

$$y = 2 - x + x^2 - \frac{1}{3}x^3 + \cdots.$$

3. Solve the initial value problem using a power series centered at $x = 0$. Write out the first four nonzero terms of the infinite series:

$$(1-x)y'' + y = 0, \quad y(0) = 3,\ y'(0) = 0.$$

Answer. Assume $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Plug into the equation

$$(1-x)y'' + y = (1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Shifting the summation index, we shall get

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty}(k+1)ka_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^k = 0,$$

thus the recurrence relation is $2a_2 + a_0 = 0$ (thus $a_2 = -\frac{1}{2}a_0$) and

$$(k+2)(k+1)a_{k+2} - (k+1)ka_{k+1} + a_k = 0, \quad k = 1, 2, \cdots .$$

The first few terms are

$$a_3 = \frac{1}{3}a_2 - \frac{a_1}{6} \Rightarrow a_3 = -\frac{1}{6}a_0 - \frac{a_1}{6},$$

$$a_4 = \frac{1}{2}a_3 - \frac{1}{12}a_2 \Rightarrow a_4 = -\frac{1}{24}a_0 - \frac{1}{12}a_1,$$

Therefore

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x - \frac{a_0}{2}x^2 - (\frac{1}{6}a_0 + \frac{1}{6}a_1)x^3 - (\frac{1}{24}a_0 + \frac{1}{12}a_1)x^4 + \cdots . \end{aligned}$$

Since $y(0) = a_0$ and $y'(0) = a_1$, we get $a_0 = 3$ and $a_1 = 0$. The first four nonzero terms of the power series solution is

$$y = 3 - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 + \cdots .$$