# Spring 2021 Math 20D Lecture B Homework #9

Due Sunday, 11:59pm, June 7th

Submit this homework through Gradescope.
**Topics covered: section 9.4, 9.5, 9.6**

1. Consider $y''' - 2y'' + y = \sin(t)$, rewrite the given scalar equation as a first-order system. Express the system in the matrix form $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$.

   **Answer.** Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$, thus

   $$x_1' = y' = x_2, \quad x_2' = y'' = x_3, \quad x_3' = y''' = 2x_3 - x_1 + \sin(t).$$

   In matrix form this is

   $$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sin(t) \end{pmatrix}$$

2. Solve the initial value problem.

   $$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

   **Answer.** Let

   $$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

   The characteristic equation of $A$ is:

   $$\det(A - rI) = \det \begin{pmatrix} 1-r & 2 \\ 3 & 2-r \end{pmatrix} = (1-r)(2-r) - 6 = 0$$

   which is $r^2 - 3r - 4 = (r+1)(r-4) = 0$. So the eigenvalues of $A$ are $r_1 = -1, r_2 = 4$.
   For $r_1 = -1$, suppose $\mathbf{v}$ is an eigenvector, then:

   $$(A - r_1 I)\mathbf{v} = \mathbf{0}$$

   which is

   $$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   We can take $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

For $r_2 = 4$, suppose $\mathbf{v}$ is an eigenvector, then:

$$(A - r_2 I)\mathbf{v} = \mathbf{0}$$

which is

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can take $\mathbf{v} = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$.

The corresponding eigensolutions are:

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} e^{4t}$$

So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} e^{4t}$$

When $t = 0$, we have $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ solve for the initial conditions, we will get $c_1 = 1, c_2 = 2$. Therefore the solution to the initial value problem is

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + 2 \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} e^{4t}.$$

3. Solve the initial value problem.

$$\mathbf{x}' = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -10 \\ -6 \end{pmatrix}$$

**Answer.** Let

$$A = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix}$$

The characteristic equation of $A$ is:

$$\det(A - rI) = \det \begin{pmatrix} 6 - r & -3 \\ 2 & 1 - r \end{pmatrix} = (6 - r)(1 - r) + 6 = 0$$

which is $r^2 - 7r + 12 = (r - 3)(r - 4) = 0$. So the eigenvalues of $A$ are $r_1 = 3, r_2 = 4$.

For $r_1 = 3$, suppose $\mathbf{v}$ is an eigenvector, then:

$$(A - r_1 I)\mathbf{v} = \mathbf{0}$$

which is

$$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can take $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $r_2 = 4$, suppose $\mathbf{v}$ is an eigenvector, then:

$$(A - r_2 I)\mathbf{v} = \mathbf{0}$$

which is

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We choose $\mathbf{v} = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$

The corresponding solutions are:

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{4t}$$

So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{4t}$$

When $t = 0$, we have $\mathbf{x}(0) = \begin{pmatrix} -10 \\ -6 \end{pmatrix}$, solve $c_1, c_2$ for the initial conditions, we will get $c_1 = 2, c_2 = -12$. Therefore the solution to the initial value problem is

$$\mathbf{x}(t) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} - 12 \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{4t}.$$

4. Find the general solution of the differential equation:

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}$$

**Answer.** Calculating the eigenvalues and eigenvectors for $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ and the two eigensolutions, we will get a fundamental matrix for the corresponding homogeneous linear system:

$$M(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

Its inverse is

$$M(t)^{-1} = \frac{1}{2}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{pmatrix}$$

The integral

$$\int M(t)^{-1}\begin{pmatrix} e^{2t} \\ 1 \end{pmatrix} dt = \frac{1}{2}\int \begin{pmatrix} 3e^{t} - e^{-t} \\ -e^{3t} + e^{t} \end{pmatrix} dt = \frac{1}{2}\begin{pmatrix} 3e^{t} + e^{-t} \\ -e^{3t}/3 + e^{t} \end{pmatrix}$$

By variation of parameters, we know the general solution for the non-homogeneous equation is:

$$M(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + M(t)\int M(t)^{-1}\begin{pmatrix} e^{2t} \\ 1 \end{pmatrix} dt$$

Which is equal to:

$$\begin{pmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 4e^{2t}/3 + 1 \\ e^{2t} + 2 \end{pmatrix}$$

where $c_1$ and $c_2$ are arbitrary constants.