7. Holomorphic Functions as conformal maps

Suppose that we take a $\mathcal{C}^1$-curve $\gamma(t)\colon [a,b] \longrightarrow U \subset \mathbb{C}$, where U is a region. Let

$$z = z(t) = x(t) + iy(t) = \gamma(t).$$

The condition that γ is $\mathcal{C}^1$ is precisely the condition that $x(t)$ and $y(t)$ are $\mathcal{C}^1$. If we are given a holomorphic function $f\colon U \longrightarrow \mathbb{C}$, then we get another $\mathcal{C}^1$-curve $w = f \circ \gamma(t)$.

Now by the chain rule $w'(t) = f'(z(t))z'(t)$. As the argument is additive when you multiply functions we see that

$$\arg w'(t_0) = \arg f'(\gamma(t_0)) + \arg z'(t_0).$$

Thus the holomorphic function $f(z)$ turns the tangent vector at $\gamma(t_0)$ through the same angle, independently of the path γ, only depending on the point $\gamma(t_0)$. In particular we see that holomorphic functions are *conformal*, that is, they preserve (oriented) angles.

If one looks at the modulus, we get a similar result,

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|.$$

Thus, in the limit, $f(z)$ changes the modulus of the distance of points from each other by the same scaling factor, regardless of the direction one chooses to approach z_0.

In this sense, as well, we can say that $f(z)$ is conformal. Moreover, it is clear that if $f(z)$ is conformal in both of these senses, then in fact it is holomorphic. Indeed, any complex number is determined by the argument and the modulus.

Perhaps surprisingly, either one of these conditions implies that f is holomorphic, assuming sufficient regularity conditions.

We first look at the condition that f preserves angles. We suppose that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous (equivalently u and v are $\mathcal{C}^1$). If this is the case then

$$w'(t_0) = \frac{\partial f}{\partial x} x'(t_0) + \frac{\partial f}{\partial y} y'(t_0).$$

We can rewrite this as

$$w'(t_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) z'(t_0) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \overline{z'(t_0)}.$$

If angles are preserved then

$$\arg(w'(t_0)/z'(t_0))$$

is independent of $z'(t_0)$. Thus

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\left(\frac{\overline{z'(t_0)}}{z'(t_0)}\right),$$

has a constant argument.

Suppose that $z'(t_0) = re^{i\theta}$. Then

$$\frac{\overline{z'(t_0)}}{z'(t_0)} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta}.$$

As $z'(t_0)$ varies, the curve above describes a circle, with radius $\frac{1}{2}|\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}|$. The argument can therefore only be constant, if

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}$$

is zero. But as have already seen these equations imply that f is holomorphic.

Now suppose that the change of scale does not depend on the direction of approach. Then the modulus of the expression above is constant. This happens on a circle, if and only if the circle has zero radius, which we have already seen implies that f is holomorphic, or the circle has centre the origin, that is

$$\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}$$

is zero. In this case f is anti-holomorphic. In this case the angle between curves is not preserved since the sign of the angle changes (for example, consider $z \longrightarrow \bar{z}$).