Math 220A Midterm Solutions

1. (a) (5 points) Let $u(x, y) = x^3 - 3xy^2 - x - y$ defined on $\mathbb{C}$. Is u harmonic on $\mathbb{C}$? Justify your answer.

   Solution: $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 1$, so $\frac{\partial^2 u}{\partial x^2} = 6x$. Meanwhile, $\frac{\partial u}{\partial y} = -6xy - 1$, and thus $\frac{\partial^2 u}{\partial y^2} = -6x$. This shows that u is harmonic on $\mathbb{C}$, as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

   (b) (5 points) Write down the Cauchy-Riemann equations.

   Solution: If u and v are two differentiable functions, the Cauchy-Riemann equations for u and v are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   (c) (5 points) If u in part (a) is harmonic, find its harmonic conjugate on $\mathbb{C}$.

   Solution: A harmonic conjugate for u is any function v such that u and v satisfy the Cauchy-Riemann equations. Thus, we must have that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 1.$$

   Integrating with respect to y, we see that

$$v = 3x^2 y - y^3 - y + C(x).$$

   To narrow down C, we use the other half of the Cauchy-Riemann equations, yielding

$$\begin{aligned} 6xy + \frac{\partial C}{\partial x} &= \frac{\partial v}{\partial x} \\ &= -\frac{\partial u}{\partial y} \\ &= 6xy + 1. \end{aligned}$$

   This means that $C(x) = x + c$ for some $c \in \mathbb{C}$, and

$$v = 3x^2 y - y^3 - y + x + c.$$

2. (a) (3 points) Write down the definition of Möbius transformations.

Solution: A function $Tz = \frac{az+b}{cz+d}$ for $z \in \mathbb{C}, a, b, c, d \in \mathbb{C}$ is a Möbius transformation if $ad - bc \neq 0$.

(b) (12 points) Let $Tz = \frac{az+b}{cz+d}$ be a Möbius transformation. Find necessary and sufficient conditions on a, b, c, d such that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Justify your answer.

Solution: See the solutions to Homework 4.

3. (a) (5 points) Let the path $\gamma(t) = e^{it}, 0 \leq t \leq \frac{\pi}{2}$. Compute $\int_\gamma ze^z dz$.

Solution: ze^z has a primitive on $\mathbb{C}$: we can find (by integration by parts, for example) that $(z-1)e^z$ is one. This allows us to use Theorem 1.18 to conclude—after calculating $\gamma(\frac{\pi}{2}) = i, \gamma(0) = 1$—that

$$\begin{aligned}\int_\gamma ze^z dz &= (i-1)e^i - (1-1)e \\ &= (i-1)e^i.\end{aligned}$$

(b) (10 points) Let the path $\gamma(t) = 2 + e^{it}, -\pi \leq t \leq \pi$. Compute $\int_\gamma (4-z^2)^{-1} dz$.

Solution: We start with the partial fraction decomposition

$$\frac{1}{4-z^2} = \frac{1}{4}(\frac{1}{2-z} + \frac{1}{2+z}).$$

Since $\operatorname{Re}\gamma(t) \geq 1$ for each t, $\frac{1}{2+z}$ has a primitive, $\log(z+2)$, in a neighborhood of the closed curve that is the image of γ. As we have seen in section 1 of chapter 4, this means that

$$\begin{aligned}\int_\gamma \frac{1}{4-z^2} dz &= \frac{1}{4}(\int_\gamma \frac{1}{2-z} dz + \int_\gamma \frac{1}{2+z} dz) \\ &= \frac{1}{4}\int_\gamma \frac{1}{2-z} dz + 0 \\ &= \frac{1}{4}\int_\gamma \frac{1}{2-z} dz.\end{aligned}$$

Although $\frac{1}{2-z}$ does not have a primitive on γ, we can still calculate its integral to be

$$\begin{aligned}\int_\gamma \frac{1}{4-z^2} dz &= \frac{1}{4}\int_{-\pi}^{\pi} \frac{1}{-e^{it}} ie^{it} dt \\ &= -\frac{i}{4}\int_{-\pi}^{\pi} dt \\ &= -\frac{i}{4}(2\pi) \\ &= -\frac{\pi i}{2}.\end{aligned}$$

4. (5 points) Let G be a region in $\mathbb{C}$ and $f : G \to \mathbb{C}$ an analytic function. Assume $|f(z)| = 1$ for all $z \in G$. Prove that f must be constant (You cannot use any result that has not been covered yet. For instance, the open mapping theorem).

Solution: Let $f = u + iv$. By assumption,

$$\begin{aligned} u^2 + v^2 &= |f|^2 \\ &= 1, \end{aligned}$$

and we can differentiate both sides with respect to x and y to find that

$$2(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}) = 2(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}) = 0.$$

In other words, on G,

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial x}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial y}. \end{aligned}$$

However, we also know that u and v satisfy the Cauchy Riemann equations, since f is analytic on the region G. Combining the two yields

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \\ &= -\frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial y} \\ &= -\frac{\partial v}{\partial y}; \end{aligned}$$

therefore, every step of this equation must be 0. As a consequence,

$$f' = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = 0$$

on the region G, so we know f must be constant.

The following question is optional and you get bonus points if you solve it.

5. (2 points) Let f be an analytic function on a region G. Assume $f' = f$ on G. Find f.

Solution: We know at least one function, e^z, has this behavior. Therefore, we consider the (analytic on G) function $h(z) := f(z)e^{-z}$. By the product rule,

$$\begin{aligned} h'(z) &= f'(z)e^{-z} - f(z)e^{-z} \\ &= f(z)e^{-z} - f(z)e^{-z} \\ &= 0. \end{aligned}$$

As in the previous problem, this means that h is some constant $c \in \mathbb{C}$, so

$$f(z) = ce^z.$$