Math 220A Midterm Solutions

1. (a) (5 points) Let \( u(x, y) = x^3 - 3xy^2 - x - y \) defined on \( \mathbb{C} \). Is \( u \) harmonic on \( \mathbb{C} \)? Justify your answer.

Solution: \( \frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 1 \), so \( \frac{\partial^2 u}{\partial x^2} = 6x \). Meanwhile, \( \frac{\partial u}{\partial y} = -6xy - 1 \), and thus \( \frac{\partial^2 u}{\partial y^2} = -6x \). This shows that \( u \) is harmonic on \( \mathbb{C} \), as

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
\]

(b) (5 points) Write down the Cauchy-Riemann equations.

Solution: If \( u \) and \( v \) are two differentiable functions, the Cauchy-Riemann equations for \( u \) and \( v \) are

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]

(c) (5 points) If \( u \) in part (a) is harmonic, find its harmonic conjugate on \( \mathbb{C} \).

Solution: A harmonic conjugate for \( u \) is any function \( v \) such that \( u \) and \( v \) satisfy the Cauchy-Riemann equations. Thus, we must have that

\[
\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 1.
\]

Integrating with respect to \( y \), we see that

\[
v = 3x^2y - y^3 - y + C(x).
\]

To narrow down \( C \), we use the other half of the Cauchy-Riemann equations, yielding

\[
6xy + \frac{\partial C}{\partial x} = \frac{\partial v}{\partial x}
\]

\[
= -\frac{\partial u}{\partial y}
\]

\[
= 6xy + 1.
\]

This means that \( C(x) = x + c \) for some \( c \in \mathbb{C} \), and

\[
v = 3x^2y - y^3 - y + x + c.
\]
2. (a) (3 points) Write down the definition of Möbius transformations.

Solution: A function $Tz = \frac{az+b}{cz+d}$ for $z \in \mathbb{C}, a, b, c, d \in \mathbb{C}$ is a Möbius transformation if $ad - bc \neq 0$.

(b) (12 points) Let $Tz = \frac{az+b}{cz+d}$ be a Möbius transformation. Find necessary and sufficient conditions on $a, b, c, d$ such that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Justify your answer.

Solution: See the solutions to Homework 4.

3. (a) (5 points) Let the path $\gamma(t) = e^{it}, 0 \leq t \leq \frac{\pi}{2}$. Compute $\int_\gamma ze^zdz$.

Solution: $ze^z$ has a primitive on $\mathbb{C}$: we can find (by integration by parts, for example) that $(z-1)e^z$ is one. This allows us to use Theorem 1.18 to conclude—after calculating $\gamma(\frac{\pi}{2}) = i, \gamma(0) = 1$—that

$$\int_\gamma ze^zdz = (i-1)e^i - (1-1)e$$

$$= (i-1)e^i.$$

(b) (10 points) Let the path $\gamma(t) = 2 + e^{it}, -\pi \leq t \leq \pi$. Compute $\int_\gamma (4 - z^2)^{-1}dz$.

Solution: We start with the partial fraction decomposition

$$\frac{1}{4 - z^2} = \frac{1}{4} \left( \frac{1}{2 - z} + \frac{1}{2 + z} \right).$$

Since $\text{Re} \gamma(t) \geq 1$ for each $t$, $\frac{1}{2 - z}$ has a primitive, $\log(z + 2)$, in a neighborhood of the closed curve that is the image of $\gamma$. As we have seen in section 1 of chapter 4, this means that

$$\int_\gamma \frac{1}{4 - z^2}dz = \frac{1}{4} \left( \int_\gamma \frac{1}{2 - z}dz + \int_\gamma \frac{1}{2 + z}dz \right)$$

$$= \frac{1}{4} \int_\gamma \frac{1}{2 - z}dz + 0$$

$$= \frac{1}{4} \int_\gamma \frac{1}{2 + z}dz.$$

Although $\frac{1}{2 - z}$ does not have a primitive on $\gamma$, we can still calculate its integral to be

$$\int_\gamma \frac{1}{4 - z^2}dz = \frac{1}{4} \int_{-\pi}^{\pi} \frac{1}{e^{it} - e^{-it}}ie^{it}dt$$

$$= -\frac{i}{4} \int_{-\pi}^{\pi} dt$$

$$= -\frac{i}{4} (2\pi)$$

$$= -\frac{\pi i}{2}.$$
4. (5 points) Let \( G \) be a region in \( \mathbb{C} \) and \( f : G \to \mathbb{C} \) an analytic function. Assume \( |f(z)| = 1 \) for all \( z \in G \). Prove that \( f \) must be constant (You cannot use any result that has not been covered yet. For instance, the open mapping theorem).

**Solution:** Let \( f = u + iv \). By assumption,

\[
\begin{align*}
  u^2 + v^2 &= |f|^2 \\
  &= 1,
\end{align*}
\]

and we can differentiate both sides with respect to \( x \) and \( y \) to find that

\[ 2\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) = 2\left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) = 0. \]

In other words, on \( G \),

\[
\begin{align*}
  \frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial x}, \\
  \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial y}.
\end{align*}
\]

However, we also know that \( u \) and \( v \) satisfy the Cauchy Riemann equations, since \( f \) is analytic on the region \( G \). Combining the two yields

\[
\begin{align*}
  \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \\
  &= -\frac{\partial v}{\partial x} \\
  &= \frac{\partial u}{\partial y} \\
  &= -\frac{\partial v}{\partial y},
\end{align*}
\]

therefore, every step of this equation must be 0. As a consequence,

\[
f' = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} = 0
\]

on the region \( G \), so we know \( f \) must be constant.
5. (2 points) Let \( f \) be an analytic function on a region \( G \). Assume \( f' = f \) on \( G \). Find \( f \).

**Solution:** We know at least one function, \( e^z \), has this behavior. Therefore, we consider the (analytic on \( G \)) function \( h(z) := f(z) e^{-z} \). By the product rule,

\[
h'(z) = f'(z)e^{-z} - f(z)e^{-z} \\
= f(z)e^{-z} - f(z)e^{-z} \\
= 0.
\]

As in the previous problem, this means that \( h \) is some constant \( c \in \mathbb{C} \), so

\[f(z) = ce^z.\]