Chapter 1

Sunday, September 2, 2018 5:24 PM

Let $\mathbb{R}$ be the set of all real numbers.

We define $\mathbb{C}$ the complex numbers to be the set of all ordered pairs (a,b) where a and b are real numbers with

addition: $(a,b)+(c,d)=(a+c,b+d)$

multiplication: $(a,b)(c,d)=(ac-bd, bc+ad)$

Remark: In this way, $\mathbb{C}$ is a field.

$(0,0)$ is the identity for addition.

$(1,0)$ is the identity for multiplication

In the futher, we will write

$$(a,b)=a+bi.$$

If $b=0$. we simply write $(a,0)=a$

Note: $i^2=(0,1)(0,1)=(-1,0)=-1$

Remark: $z^2+1=0$ has a root in $\mathbb{C}$.

Important properties:

Let x,y,z be complex numbers.

(1) Associativity: $(x+y)+z=x+(y+z)$

$x\cdot(y\cdot z)=(x\cdot y)\cdot z$

(2) Commutativity: $x+y=y+x$

$x\cdot y=y\cdot x$

(3) Additive Inverse: Every $z\in\mathbb{C}$ has an additive

inverse, denoted $-z$, such that

(3) Additive Inverse: Every $z \in \mathbb{C}$ has an additive inverse, denoted $-z$, such that

$$z + (-z) = 0$$

(4) Multiplicative Inverse: Every $z \neq 0 \in \mathbb{C}$ has an multiplicative inverse, denoted z^{-1} or $\frac{1}{z}$, such that $z \cdot \frac{1}{z} = 1$.

(5) Distributivity:

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Definition: Let $z \in \mathbb{C}$. Write $z = a + ib$ where $a, b \in \mathbb{R}$. Then we call

a: Real part of z

b: Imaginary part of z

Notation: $a = \operatorname{Re} z$

$b = \operatorname{Im} z$

Definition: Let $z \in \mathbb{C}$. Write $z = a + ib$, where $a, b \in \mathbb{R}$. We define

$|z| = (a^2 + b^2)^{\frac{1}{2}}$ to be the absolute value of z;

define $\bar{z} = a - bi$ to be the conjugate of z.

Excercise: Let $a, b \in \mathbb{R}$. Not both zero.

Excercise: Let $a, b \in \mathbb{R}$. Not both zero.

Set $z = \frac{1}{a+ib}$

(1) Find $\operatorname{Re} z$, $\operatorname{Im} z$

(2) Find $|z|$, $\bar{z}$

Solution:

$$z = \frac{1}{a+ib} = \frac{1}{a+ib} \frac{a-ib}{a-ib}$$

$$= \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

Thus

(1) $\operatorname{Re} z = \frac{a}{a^2+b^2}$, $\operatorname{Im} z = \frac{-b}{a^2+b^2}$

(2) $\bar{z} = \frac{a}{a^2+b^2} + i\frac{b}{a^2+b^2}$

$$|z| = \left(\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{b}{a^2+b^2}\right)^2\right)^{1/2}$$

$$= \left(\frac{a^2+b^2}{(a^2+b^2)^2}\right)^{1/2}$$

$$= \frac{1}{\sqrt{a^2+b^2}}$$

Important properties:

(1) $\operatorname{Re} z = \frac{1}{2}(z+\bar{z})$; $\operatorname{Im} z = \frac{1}{2i}(z-\bar{z})$

(2) $\overline{(z+w)} = \bar{z} + \bar{w}$; $\overline{zw} = \bar{z}\bar{w}$

(3) $|zw| = |z||w|$.

(4) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$, where $w \neq 0$.

(5) $|\bar{z}| = |z|$

(6) $|z|^2 = z\bar{z}$. Thus if $z \neq 0$, then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

(5) $|\bar{z}| = |z|$

(6) $|z|^2 = z\bar{z}$. Thus if $z \neq 0$, then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

Excercise: Let $w = \dfrac{z-a}{z+a}$ where $z \in \mathbb{C}, a \in \mathbb{R}$.

Find $\operatorname{Re} w$, $\operatorname{Im} w$.

Solution:

$$w = \frac{z-a}{z+a} \, \frac{\bar{z}+\bar{a}}{\bar{z}+\bar{a}}$$

$$= \frac{z-a}{z+a} \, \frac{\bar{z}+a}{\bar{z}+a}$$

$$= \frac{|z|^2 + az - a\bar{z} - a^2}{|z|^2 + az + a\bar{z} + a^2}$$

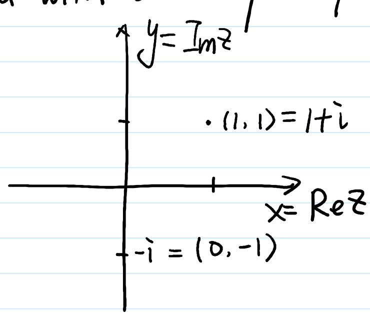
$$= \frac{|z|^2 - a^2 + a(2i \operatorname{Im} z)}{|z|^2 + a^2 + a(2\operatorname{Re} z)}$$

$$\Rightarrow \operatorname{Re} w = \frac{|z|^2 - a^2}{|z|^2 + a^2 + 2a\operatorname{Re} z}$$

$$\operatorname{Im} w = \frac{2a \operatorname{Im} z}{|z|^2 + a^2 + 2a\operatorname{Re} z}$$

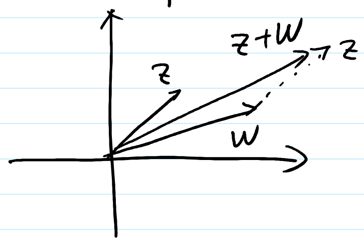
The complex plane

Every complex number $z \in \mathbb{C}$ can be identified with the unique point $(\operatorname{Re} z, \operatorname{Im} z) \in \mathbb{R}^2$



Note: The addition of complex numbers is precisely

Note: The addition of complex numbers is precisely the addition law of the vector space $\mathbb{R}^2$.



Remark: $|z-w|$ is exactly the distance between z and w.

Proposition:

(1) $|z_1-z_2| \le |z_1-z_3| + |z_3-z_2|$

(2) $|z+w| \le |z| + |w|$

(3) $\big||z|-|w|\big| \le |z-w|$

Pf: We will only prove (2).

$$|z+w|^2 = |z|^2 + 2\mathrm{Re}(z\bar{w}) + |w|^2$$

Note for every $\eta \in \mathbb{C}$, $\mathrm{Re}\,\eta \le |\eta|$

Thus
$$|z+w|^2 \le |z|^2 + 2|z||w| + |w|^2$$
$$\le (|z|+|w|)^2$$

Hence $|z+w| \le |z|+|w|$.

Q: When the "=" holds in (2)?

A: The "=" holds iff

$$\mathrm{Re}\, z\bar{w} = |z|\,|w| = |z\bar{w}|$$

This is equivalent to say

$$z\bar{w} \in \mathbb{R} \text{ and } z\bar{w} \ge 0$$

We have two cases:

① If $w \ne 0$, then

$$z\bar{w} \ge 0$$

$\Leftrightarrow \quad z\bar{w}\frac{w}{w} \geq 0$

$\Leftrightarrow \quad \frac{z}{w}|w|^2 \geq 0$

$\Leftrightarrow \quad \frac{z}{w} \geq 0$

$\Leftrightarrow \quad z = tw$ for some $t \geq 0$

② If $w=0$, always true.

Hence, the "=" holds iff

$z = tw$ for some $t \geq 0$ or $w = 0$.

Corollary: $|z_1+z_2+\cdots+z_n| \leq |z_1|+|z_2|+\cdots+|z_n|$

Pf: prove by induction and apply (2).

4. Polar representation and roots of complex numbers

Let $z = x+iy$ be a pt in the complex plane $\mathbb{C}$.

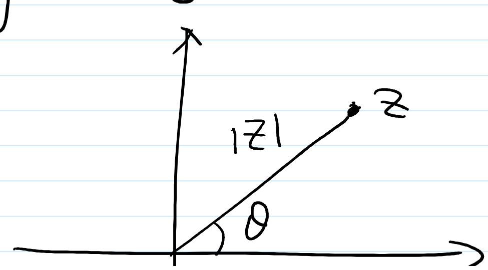
The pt z has polar coordinates (r, θ):

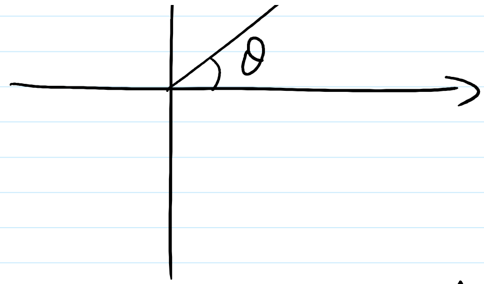
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Note: $r = |z|$

θ: The angle between the positive real axis and the line segment from 0 to z.

θ is called the argument of z, denoted by $\theta = \arg z$.





Note: θ is only well-define up to a multiple of 2π.

Definition:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Note: In the textbook, $e^{i\theta}$ is written as $\operatorname{cis}\theta$

Remark: Eevery $z \in \mathbb{C}$ can be written as

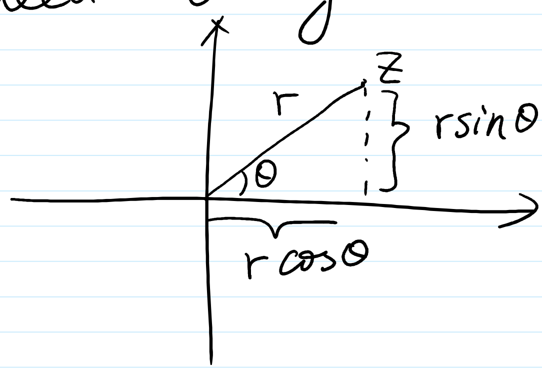
$$z = re^{i\theta}$$

where $r \geq 0$, $\theta \in \mathbb{R}$

r is unique and $r = |z|$

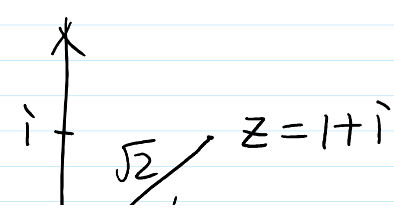
θ is unique to a multiple of 2π

Indeed. $\theta = \arg\theta$



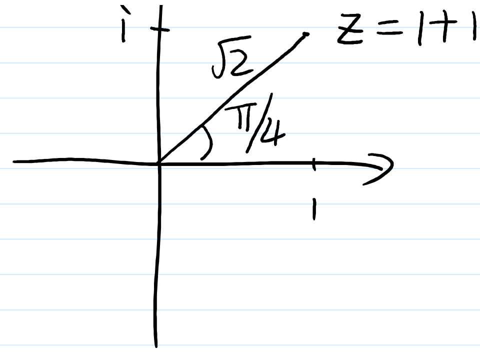
Example: (1) $1 = e^{i0} = e^{i2\pi} = \cdots$

(2) $1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$



(2) $1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$= \sqrt{2}\, e^{i\frac{\pi}{4}} = \cdots$



Proposition:

(1) Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

Then $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

(2) Let $z_j = r_j e^{i\theta_j}$, $1 \le j \le n$

Then

$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n e^{i(\theta_1 + \cdots + \theta_n)}$

Pf: (1)

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i\theta_1} e^{i\theta_2} \\ &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \big((\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_1) + i(\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1) \big) \\ &= r_1 r_2 \big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \big) \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} \end{aligned}$$

(2) prove by induction and apply (1).

Corollary: Let $z = re^{i\theta}$.

Then $z^n = r^n e^{in\theta}$

de Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Excercise: Find all $z \in \mathbb{C}^n$ such that

$$z^n = a.$$

where $a = |a| e^{i\alpha}$, $|a| > 0$.

Solution: $$z^n = |a| e^{i\alpha}$$

$$\Rightarrow \quad z = |a|^{\frac{1}{n}} e^{i\frac{\alpha + 2\pi k}{n}}, \quad 0 \le k \le n-1.$$

Q: why k only runs to $n-1$?

Example: Solve $z^n = 1$

Solution: Note $1 = e^{i \cdot 0}$.

Thus $z = 1, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \cdots, e^{i\frac{2\pi(n-1)}{n}}$

In particular, the cube roots of unity are:

($n = 3$)

$$z = 1$$

$$z = e^{i\frac{2\pi}{3}} = \frac{1}{2}(-1 + i\sqrt{3})$$

$$z = e^{i\frac{4\pi}{3}} = \frac{1}{2}(-1 - i\sqrt{3})$$

Example: Let $z = e^{i\frac{2\pi}{n}}$.

prove $1 + z + \cdots + z^{n-1} = 0$

proof: Note $z^n - 1 = 0$

But $(z^n - 1) = (z-1)(z^{n-1} + z^{n-2} + \cdots + 1)$,

and $z \neq 1$ i.e., $z - 1 \neq 0$

$\Rightarrow z^{n-1} + z^{n-2} + \cdots + 1 = 0$.