Metric Space and the Topology of $\mathbb{C}$

1. Definition and examples of metric spaces

Defn: A metric space is a pair (X,d) where X is a set and d is a function from $X \times X$ in $\mathbb{R}$, called a distance function or metric, which satisfies the following conditions for x, y and z in X.

① $d(x,y) \geq 0$

② $d(x,y) = 0$ iff $x=y$

③ $d(x,y) = d(y,x)$

④ $d(x,z) \leq d(x,y) + d(y,z)$

--- triangle inequality

Defn: Let (X,d) be a metric space.

Fix $x \in X$ and $r > 0$.

Define $B(x;r) = \{y \in X : d(x,y) < r\}$

$\bar{B}(x;r) = \{y \in X : d(x,y) \leq r\}$

$B(x;r)$ and $\bar{B}(x;r)$ are called the open and closed balls with center x and radius r.

Example:

1. Let $X = \mathbb{R}$ or $\mathbb{C}$ and define $d(z,w) = |z-w|$.

Then $(\mathbb{R}, d)$ and $(\mathbb{C}, d)$ are metric spaces.

2. Let $X = \mathbb{C}$ and define
$$d(x+iy, a+ib) = |x-a| + |y-b|.$$
Then $(\mathbb{C}, d)$ is a metric space.

3. Let $X = \mathbb{C}$ and define
$$d(x+iy, a+ib) = \max\{|x-a|, |y-b|\}.$$

4. Let $X = \mathbb{R}^n$ and for $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{R}^n$ define
$$d(x,y) = \left[\sum_{j=1}^{n} |x_j - y_j|^2\right]^{\frac{1}{2}}$$

5. Let $X = \mathbb{C}^n$ and for $z = (z_1, \cdots, z_n), w = (w_1, \cdots, w_n) \in \mathbb{C}^n$ define
$$d(z,w) = \left[\sum_{j=1}^{n} |z_j - w_j|^2\right]^{\frac{1}{2}}.$$

6. Let S be any set and $B(S)$ the set of all functions $f: S \to \mathbb{C}$ such that
$$\|f\|_\infty \equiv \sup\{|f(s)| : s \in S\} < \infty$$
Define $d(f,g) = \|f-g\|_\infty$

Verify (S, d) is a metric space.

<u>Defn</u>: Let (X,d) be a metric space. A set $G \subset X$ is open if for each $x \in G$, $\exists\ \varepsilon > 0$, s.t
$$B(x, \varepsilon) \subset G$$

Remark: We will write ϕ for the empty set. ϕ is also an open set.

Proposition: Let (X, d) be a metric space. Then

(a) The set X and ϕ are open.

(b) If $G_1, \cdots, G_n$ are open, then so is $\bigcap_{k=1}^{n} G_k$

(c) If $\{G_j : j \in J\}$ is a collection of open sets in X, J any indexing set, then $G = \bigcup_{j \in J} G_j$ is also open.

Defn: A set $F \subset X$ is closed if its complement, $X - F$, is open.

Proposition: Let (X, d) be a metric space. Then

(a) The sets X and ϕ are closed.

(b) If $F_1, \cdots, F_n$ are closed sets in X, then so is $\bigcup_{k=1}^{n} F_k$.

(c) If $\{F_j : j \in J\}$ is any collection of closed sets in X, J any indexing set, then $F = \bigcap_{j \in J} F_j$ is also closed.

Defn: Let A be a subset of X. Then the interior of A, $\operatorname{int} A$, is the set $\bigcup \{G : G \text{ is open and } G \subset A\}$.

The closure of A, A^-, is the set

$$\bigcap \{F : F \text{ is closed and } F \supset A\}$$

The boundary of A is denoted by ∂A and defined by $\partial A = A^- \cap (X - A)^-$

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Remark: $\operatorname{int} A$ is always open

A^- is always closed.

∂A is always closed

Example: Let $A = \{a+bi : a, b \in \mathbb{Q}\} \subset \mathbb{C}$

Then $\operatorname{int} A = \phi$, and $A^- = \mathbb{C}$

Defn: A subset A of a metric space is dense if $A^- = X$

Example: $A = \{a+bi : a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$

proposition: Let A and B be subsets of a metric space (X, d). Then

(a) A is open iff $A = \operatorname{int} A$.

(b) A is closed iff $A = A^-$.

(c) $\operatorname{int} A = X - (X-A)^-$;

$A^- = X - \operatorname{int}(X-A)$

$\partial A = A^- - \operatorname{int} A$

(d) $(A \cup B)^- = A^- \cup B^-$

(e) $x_0 \in \operatorname{int} A$ iff $\exists\ \varepsilon > 0$ s.t

$B(x_0; \varepsilon) \subset A$

(f) $x_0 \in A^-$ iff for every $\varepsilon > 0$,

$B(x_0; \varepsilon) \cap A \neq \phi$

Example: Let $X = \mathbb{R}$ be equipped with the standard metric $d(x, y) = |x-y|$

Example: Let $X=\mathbb{R}$ be equipped with the standard metric $d(x,y)=|x-y|$.

Let $A=(0,1]$. Then

$$X-A=(-\infty,0]\cup(1,+\infty)$$
$$\operatorname{int}A=(0,1)$$
$$\bar{A}=[0,1]$$
$$\operatorname{int}(X-A)=(-\infty,0)\cup(1,+\infty)$$
$$\overline{(X-A)}=(-\infty,0]\cup[1,+\infty)$$
$$\partial A=\{0,1\}$$

Q: Let $A\subset\mathbb{R}$. Assume $\partial A=\phi$. Find A.

In the further, when we discuss $\mathbb{R}$ or $\mathbb{C}$, unless otherwise stated, we will always assume it has the metric $d(z,w)=|z-w|$.

2. Connectedness

Defn: A metric space (X,d) is connected iff the only subsets of X which are open and closed are $\emptyset$ and X. Let $A\subset X$. We say A is a connected subset of X if the metric space (A,d) is connected.

Proposition: A set $X\subset\mathbb{R}$ is connected iff X is an interval.

Pf: 1. "$\Leftarrow$"

Let $X=[a,b]$ where $a,b\in\mathbb{R}$.

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Let $X = [a, b]$. where $a, b \in \mathbb{R}$.

($X = (a,b), (a,b], [a,b), (-\infty, b)$, etc can be proved similarly)

We will prove by contradiction. Suppose X is Not connected. Then $\exists$ an subset $A \subset X$ s.t

① $A \neq \phi$, $A \neq X$.

② A is open and closed in X.

We can assume $a \in A$ (why?)

Since A is open, and $a \in A$, there is an $\varepsilon > 0$ s.t $[a, a+\varepsilon) \subset A$. Let

$$r = \sup\{\varepsilon : [a, a+\varepsilon) \subset A\}$$

Claim: $[a, a+r) \subset A$.

Pf: Fix any $x \in [a, a+r)$. We need to show $x \in A$.

By the definition of r, there exists ε_0 s.t $\varepsilon_0 > x - a$ and $[a, a+\varepsilon_0) \subset A$.

Then $x \in [a, a+\varepsilon_0) \subset A$.

Claim: $a+r \notin A$. Thus $a+r \in X - A$.

Pf: If, on the contrary, $a+r \in A$, then by the openness of A $\exists \delta > 0$ s.t $[a+r, a+r+\delta) \subset A$

Pf: If, on the contrary, $a+r \in A$, then by the openness of A, $\exists\, \delta > 0$ s.t $[a+r, a+r+\delta) \subset A$. But this gives $[a, a+r+\delta) \subset A$, contradicting the definition of r.

Now $a+r \in X-A$ and $X-A$ is open (why?). Then $\exists\, \delta > 0$ s.t $(a+r-\delta, a+r] \subset X-A$, contradicting the first claim.

2. "$\Rightarrow$" we will need the lemma.

Lemma: A set $U \subset \mathbb{R}$ is an interval iff for any two pts $a, b \in U$ with $a < b$, the interval $[a,b] \subset U$.

Pf: Excercise.

By the lemma, to prove X is an interval, we just need to show for any $a, b \in \mathbb{R}$, $a < b$ it holds that $[a,b] \subset X$. Suppose NOT. $\exists\, a < c < b$ s.t $c \notin X$. Then let

$$X_1 = \{x \in X ; \ x < c\}$$

Then $X_1 \neq \phi$ as $a \in X_1$; $X_1 \neq X$ as $b \notin X$.

Claim: X_1 is open and closed in X.

This contradicts with the assumption that X is

Claim: X_1 is open and closed in X.

This contradicts with the assumption that X is connected. Hence we must have $[a,b] \subset X$.

Defn: A subset D of a metrix space (X,d) is a component of X if it is a maximal connected subset of X. That is, D is connected and there is no connected subset of X that properly contains D.

Example: Let $X = (-1,1) \cup (2,3)$

Then $(-1,1)$ and $(2,3)$ are two components

Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots,\}$

Then each pt in X is a component

Note $\{\frac{1}{n}\}$, $n \geq 1$, is open in X;

while $\{0\}$ is NOT open in X.

Lemma: Let $x_0 \in X$ and let $\{D_j : j \in J\}$ be a collection of connected subsets of X such that $x_0 \in D_j$ for each $j \in J$.

Then $D = \bigcup_{j \in J} D_j$ is connected.

Pf: Let A be a subset of the metric space (D,d) which is both open and closed. Suppose $A \neq \emptyset$.

We will prove A must be D.

Note $A \cap D_j$ is both open and closed in D_j (Why?)

Since each D_j is connected, $A \cap D_j$ is either $\emptyset$ or D_j.

Since $D = \bigcup_{j \in J} D_j$ and $A \subset D$ is not empty, there exists some k s.t $A \cap D_k \neq \phi$. As D_k is connected we must have $A \cap D_k = D_k$. Thus $D_k \subset A$. In particular, $x_0 \in A$. Hence $x_0 \in A \cap D_j$ for all j, and $A \cap D_j = \phi$. Thus $A \cap D_j = D_j$ for all j. We conclude $D = A$, so thus D is connected.

Thm. Let (X, d) is a metric space. Then

(a) Each x_0 in X is contained in a component of X.

(b) Distinct components of X are disjoint.

Pf: Page 16 in the book.

proposition: (a) If $A \subset X$ is connected and $A \subset B \subset A^-$, then B is connected. In particular, A^- is connected.

(b) If C is a component of X then C is closed

Pf: (a) We prove by contradiction. Suppose B is NOT connected. Then $\exists B_1, B_2 \subset B$, $B_1 \neq \phi$ $B_2 \neq \phi$. both open in B.

$$B_1 \cup B_2 = B \, ; \, B_1 \cap B_2 = \phi.$$

Write $A_1 = B_1 \cap A$. $A_2 = B_2 \cap A$

Since $A \subset B$, $\Rightarrow$ $A_1 \cup A_2 = A$

Note A_1, A_2 are open in A (why?)
$A_1 \cap A_2 = \phi$.

Claim: $A_1 \neq \phi$ and $A_2 \neq \phi$.

$A_1 \cap A_2 = \phi$.

Claim: $A_1 \neq \phi$ and $A_2 \neq \phi$.

Pf: Suppose, say $A_1 = \phi$. Then $A_2 = A$

But $A_2 \subset B_2 \Rightarrow A_2^- = A^- \subset B_2^- \subset B^- \subset A^-$

This implies $B_2^- = B^-$. But B_2 is closed in B

we must have $B_2 = B$. Then $B_1 = \phi$.

This contradicts the assumption on B_1, B_2.

The claim contradicts the connected-ness of A.

(b). By (a), C^- is also connected, and note $C \subset C^-$. But C is a maximal connected subset, we must have $C = C^-$. This implies, C is closed.

Thm. Let G be open in $\mathbb{C}$; then the components of G are open and there are only a countable number of them.

Pf: Let C be a component of G and let $x_0 \in C$. Since G is open there is an $\varepsilon > 0$ with $B(x_0; \varepsilon) \subset G$.

By the lemma above, $B(x_0; \varepsilon) \cup C$ is connected, so it must be C. That is, $B(x_0, \varepsilon) \subset C$ and C is, therefore, open.

To see that the number of components is countable, let $S = \{a + ib : a, b \in \mathbb{Q} \text{ and } a + bi \in G\}$. Then S is countable. Moreover, every $z \in S$ is contained in one (and only one) component of G, and each component of G contains a pt of S. So the number of component is countable.