3. Sequences and completeness

<u>Defn</u>: If $\{x_1, x_2, \cdots\}$ is a sequence in a metric space (X, d) then $\{x_n\}$ converges to x. if for every $\varepsilon > 0$, there is an integer N such that $d(x, x_n) < \varepsilon$ whenever $n \geq N$.

Notation: $x = \lim x_n$ or $x_n \to x$

Remark: $x = \lim x_n$ if $0 = \lim d(x, x_n)$

If $X = \mathbb{C}$ then $z = \lim z_n$ means that for each $\varepsilon > 0$ there is an N s.t $|z - z_n| < \varepsilon$ when $n \geq N$.

proposition 3.2: A set $F \subset X$ is closed iff for each sequence $\{x_n\}$ in F with $x = \lim x_n$ we have $x \in F$.

Pf: ① "$\Rightarrow$"

Suppose F is closed and $x = \lim x_n$ where each x_n is in F. So for every $\varepsilon > 0$, there is a pt $x_n \in B(x, \varepsilon)$; that is $B(x; \varepsilon) \cap F = \phi$. So that $x \in F^- = F$

② "$\Leftarrow$"

We will prove by contrapositive.
Now suppose F is not closed; so $\exists$ a pt $x_0 \in F^-$ which is not F. since $x_0 \in F^-$ for $\forall \varepsilon > 0$, we have $B(x_0; \varepsilon) \cap F \neq \phi$. In particular. for every integer n there is a pt x_n in $B(x_0; \frac{1}{n}) \cap F$.
Thus $d(x_0, x_n) < \frac{1}{n}$ which implies that $x_n \to x_0$.

<u>Defn</u>. If $A \subset X$ then a pt $x \in X$ is a limit pt of A if there is a sequence $\{x_n\}$ of distinct pts in A such that $x = \lim x_n$.

proposition 3.4. (a) A set is closed iff it contains all its limit pts

proposition 3.4. (a) A set is closed iff it contains all its limit pts

(b) If $A \subset X$ then $\bar{A} = A \cup \{\text{all limit pts of } A\}$

Defn: A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if for every $\varepsilon > 0$ there is an integer N s.t $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Defn. A metric space is complete if every Cauchy sequence has a limit in X.

Proposition: $\mathbb{C}$ is complete.

If $A \subset X$, we define the diamoter of A by

$$\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}$$

Thm 3.7 (Cantor's Theorem) A metric space (X, d) is complete iff for any sequence $\{F_n\}$ of non-empty closed sets with $F_1 \supset F_2 \supset \cdots$ and $\operatorname{diam} F_n \to 0$. $\bigcap_{n=1}^{\infty} F_n$ consists of a single pt.

Pf: See Page 19 in the book.

Proposition 3.8: Let (X, d) be a complete metric space and let $Y \subset X$. Then (Y, d) is a complete metric space iff Y is closed in X.

Pf: ① "$\Leftarrow$"

Assume Y is closed. For any Cauchy Sequence $\{y_n\} \subset Y \subset X$, as X is complete. $\exists x_0 \in X$ s.t $y_n \to x_0$. Thus x_0 is a limit pt of Y. As Y is closed. we have $x_0 \in Y$. We conclude Y is complete

② "$\Rightarrow$"

Assume Y is complete. Let x_0 be a limit pt of Y. Then there is a sequence $\{y_n\}$ of pts in Y s.t

$$x_0 = \lim y_n$$

Hence $\{y_n\}$ is a Cauchy sequence (why?)

$x_0 = \lim y_n$

Hence $\{y_n\}$ is a Cauchy sequence (why?) and must converge to a pt $y_0 \in Y$, as Y is complete. This implies $y_0 = x_0$ and so Y contains all its limit pts. Hence Y is closed.

4. Compactness.

Defn: (1) A subset K of a metric space X is compact if for every collection $\mathcal{G}$ of open sets in X that covers K, i.e.,

$$K \subset \bigcup \{G : G \in \mathcal{G}\} \quad (*)$$

there is a finite number of sets $G_1, \cdots, G_n$ in $\mathcal{G}$ such that $K \subset G_1 \cup G_2 \cup \cdots \cup G_n$.

(2) A collection of set $\mathcal{G}$ satisfying $(*)$ is called a cover of K; if each member of $\mathcal{G}$ is an open set it is called an open cover of K; if $\mathcal{G}$ has finitely many elements it is called a finite cover of K.

Example: (1) $\mathbb{Z} \subset \mathbb{R}$ is Not compact

(2) $(0,1) \subset \mathbb{R}$ is NOT compact. Indeed. Let $G_n = (0, 1-\frac{1}{n})$; $n \geq 2$ Then

$$\mathcal{G} = \{G_n : n \geq 2\}$$

is an open cover of $(0,1)$. One can NOT find a finite cover from $\mathcal{G}$.

Proposition 4.3 Let K be a compact subset of X; then (a) K is closed

(b) If F is closed and $F \subset K$ then F is compact.

If $\mathcal{F}$ is a collection of subsets of X, we say that $\mathcal{F}$ has finite intersection property (f.i.p) if whenever $\{F_1, \cdots, F_n\} \subset \mathcal{F}$, $F_1 \cap F_2 \cap \cdots \cap F_n = \phi$.

Example: Let $X = (0,1)$. $F_n = (0, \frac{1}{n}]$, $n \geq 2$. Then $\mathcal{F} = \{F_n : n \geq 2\}$ has f.i.p. But

$$\cap \{F : F \in \mathcal{F}\} = \phi$$

$\mathcal{F} = \{F_n : n \geq 2\}$ has f.i.p. But
$\bigcap \{F : F \in \mathcal{F}\} = \phi$.

proposition 4.4: A set $K \subset X$ is compact iff every collection $\mathcal{F}$ of closed subsets of K with f.i.p satisfies $\bigcap \{F : F \in \mathcal{F}\} \neq \phi$

Pf: See P_{21} in the book.

Corollary 4.5: Every compact metric space is complete.

Corollary 4.6: If X is compact, then every infinite set has a limit pt in X.

pf: Let S be an infinite subset of X and suppose S has no limit pts. Let $\{a_1, a_2, \cdots\}$ be a sequence of distinct pts in S; then

$F_n = \{a_n, a_{n+1}, \cdots\}$ also has no limit pts.

This implies F_n is closed.

Note $\mathcal{F} = \{F_n : n \geq 1\}$ has f.i.p

However, since $a_1, a_2, \cdots$ are distinct, $\bigcap_{n=1}^{\infty} F_n = \phi$, contradicting proposition 4.5

Defn: A metric space (X, d) is sequentially compact if every sequence in X has a convergent subsequence.

Lebesgue's Covering Lemma

If (X, d) is sequentially compact and $\mathcal{G}$ is an open cover of X then there is an $\varepsilon > 0$ s.t if x is in X, there is an set $G \in \mathcal{G}$ with $B(x; \varepsilon) \subset G$.

Pf: We prove by contradiction. Suppose $\mathcal{G}$ is an open cover of X and no such $\varepsilon > 0$ can be found. In particular, for every $n \in \mathbb{Z}^+$ there is a pt $x_n \in X$ such that $B(x_n, \frac{1}{n})$ is NOT contained in any set G in $\mathcal{G}$. Since X is sequentially compact there is a pt $x_0 \in X$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t $x_0 = \lim x_{n_k}$. Let $G_0 \in \mathcal{G}$ s.t $x_0 \in G_0$. There $\exists\ \varepsilon > 0$ s.t $B(x_0, \varepsilon) \subset G_0$. Now let

$\{x_{n_k}\}$ of $\{x_n\}$ s.t $x_0 = \lim x_{n_k}$. Let $G_0 \in \mathcal{G}$ s.t $x_0 \in G_0$. There $\exists\ \varepsilon > 0$ s.t $B(x_0, \varepsilon) \subset G_0$. Now let N be such that $d(x_0, x_{n_k}) < \varepsilon/2$ for all $n_k \geq N$. Let $n_k \geq \max\{N, \frac{2}{\varepsilon}\}$, let $y \in B(x_{n_k}, \frac{1}{n_k})$. Then

$$d(x_0, y) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y) < \varepsilon/2 + \frac{1}{n_k} < \varepsilon.$$

That is, $B(x_{n_k}; \frac{1}{n_k}) \subset B(x_0; \varepsilon) \subset G_0$, contradicting to the choice of x_{n_k}.

Thm 4.9. Let (X, d) be a metric space. Then the following are equivalent (TFAE):

(a) X is compact;

(b) Every infinite set in X has a limit pt.

(c) X is sequentially compact

(d) X is complete and for every $\varepsilon > 0$ there are a finite number of pts $x_1, \cdots, x_n$ in X s.t $X = \bigcup_{k=1}^{n} B(x_k; \varepsilon)$.

(The property in (d) is called total boundedness)

Pf: See P22 in the book.

Theorem 4.10 (Heine-Borel Theorem)

A subset K of $\mathbb{R}^n$ $(n \geq 1)$ is compact iff K is closed and bounded.

Pf: Page 23 in book.