3. Sequences and completeness

**Definition:** If \( \{x_1, x_2, \ldots\} \) is a sequence in a metric space \((X, d)\) then \( \{x_n\} \) converges to \( x \).

If for every \( \varepsilon > 0 \), there is an integer \( N \) such that \( d(x, x_n) < \varepsilon \) whenever \( n \geq N \).

**Notation:** \( x = \lim x_n \) or \( x_n \to x \)

**Remark:** \( x = \lim x_n \) if \( 0 = \lim d(x, x_n) \)

If \( X = \mathbb{C} \) then \( z = \lim z_n \) means that for each \( \varepsilon > 0 \), there is an \( N \) such that \( |z - z_n| < \varepsilon \) when \( n \geq N \).

**Proposition 3.2:** A set \( F \subset X \) is closed if and only if for each sequence \( \{x_n\} \) in \( F \) with \( x = \lim x_n \) we have \( x \in F \).

**Pf:**

1. \( \Rightarrow \)

Suppose \( F \) is closed and \( x = \lim x_n \) where each \( x_n \) is in \( F \). So for every \( \varepsilon > 0 \), there is a pt \( x_0 \in B(x, \varepsilon) \); that is \( B(x, \varepsilon) \cap F = \emptyset \).

So that \( x \notin F^- = F \)

2. \( \Leftarrow \)

We will prove by contrapositive.

Now suppose \( F \) is not closed; so \( \exists \) a pt \( x_0 \in F^- \) which is not in \( F \). Since \( x_0 \in F^- \), for all \( \varepsilon > 0 \), we have \( B(x_0, \varepsilon) \cap F = \emptyset \). In particular, for every integer \( n \) there is a pt \( x_n \) in \( B(x_0, \varepsilon) \cap F \).

Thus \( d(x_0, x_n) < \frac{\varepsilon}{2} \) which implies that \( x_n \to x_0 \).

**Definition:** If \( A \subset X \) then a pt \( x \in X \) is a limit pt of \( A \) if there is a sequence \( \{x_n\} \) of distinct pts in \( A \) such that \( x = \lim x_n \).

**Proposition 3.4:** (a) A set is closed iff it contains all its limit pts
Proposition 3.4: (a) A set is closed iff it contains all its limit pts.
(b) If \( A \subseteq X \) then \( A = A \cup \{ \text{all limit pts of } A \} \).

Defn: A sequence \( \{x_n\} \subseteq X \) is called a Cauchy sequence if for every \( \varepsilon > 0 \) there is an integer \( N \) s.t. \( d(x_n, x_m) < \varepsilon \) for all \( n, m \geq N \).

Defn: A metric space is complete if every Cauchy sequence has a limit in \( X \).

Proposition: \( \mathbb{C} \) is complete.

If \( A \subseteq X \), we define the diameter of \( A \) by:
\[
\text{diam } A = \sup \{ d(x, y) : x, y \in A \}
\]

Thm 3.7 (Cantor's Theorem) A metric space \( (X, d) \) is complete iff for any sequence \( \{F_n\} \) of non-empty closed sets with \( F_1 \supseteq F_2 \supseteq \ldots \) and \( \text{diam } F_n \to 0 \), \( \bigcap_{n=1}^{\infty} F_n \) consists of a single pt.

Proof: See Page 19 in the book.

Proposition 3.8: Let \( (X, d) \) be a complete metric space and let \( Y \subseteq X \). Then \( (Y, d) \) is a complete metric space iff \( Y \) is closed in \( X \).

Proof:

1. \( \Rightarrow \)
   Assume \( Y \) is closed. For any Cauchy sequence \( \{y_n\} \subseteq Y \subseteq X \), as \( X \) is complete, \( \exists x_0 \in X \) s.t. \( y_n \to x_0 \). Thus, \( x_0 \) is a limit pt of \( Y \). As \( Y \) is closed, we have \( x_0 \in Y \). We conclude \( Y \) is complete.

2. \( \Leftarrow \)
   Assume \( Y \) is complete. Let \( x_0 \) be a limit pt of \( Y \).
   Then there is a sequence \( \{y_n\} \) of pts in \( Y \) s.t. \( x_0 = \lim y_n \).
   Hence \( \{y_n\} \) is a Cauchy sequence (why?)
\[ x_0 = \lim y_n \]

Hence \( \{ y_n \} \) is a Cauchy sequence (why?)
and must converge to a pt \( y_0 \in Y \), as \( Y \)
is complete. This implies \( y_0 = x_0 \) and so \( Y \) contains all its limit pts. Hence \( Y \) is closed.

4. Compactness.

Defn. (1) A subset \( K \) of a metric space \( X \) is compact if for every collection \( \mathcal{G} \) of open sets in \( X \) that covers \( K \), i.e.,
\[ K \subseteq \bigcup \{ G : G \in \mathcal{G} \} \quad \text{(1)} \]
there is a finite number of sets \( G_1, \ldots, G_n \).
In \( \mathcal{G} \) such that \( K \subseteq G_1 \cup G_2 \cup \ldots \cup G_n \).

(2) A collection of sets \( \mathcal{G} \) satisfying (1) is called a cover of \( K \); if each member of \( \mathcal{G} \) is an open set it is called an open cover of \( K \);
if \( \mathcal{G} \) has finitely many elements it is called a finite cover of \( K \).

Example: (1) \( \mathbb{Z} \subset \mathbb{R} \) is not compact.

(2) \( (0, 1) \subset \mathbb{R} \) is NOT compact. Indeed.
Let \( G_n = (0, 1 - \frac{1}{n}) ; \ n \geq 2 \) Then
\[ \mathcal{G} = \{ G_n : n \geq 2 \} \]
is an open cover of \( (0, 1) \). One can find a finite cover from \( \mathcal{G} \).

Proposition 4.3 Let \( K \) be a compact subset of \( X \);
then (a) \( K \) is closed
(b) If \( F \) is closed and \( F \cap K \)
then \( F \) is compact.

If \( \mathcal{J} \) is a collection of subsets of \( X \), we say that \( \mathcal{J} \) has finite intersection property (f.i.p.)
if whenever \( \{ F_1, \ldots, F_n \} \subset \mathcal{J} \) \( \bigcap F_1 \cap \ldots \cap F_n \neq \emptyset \).

Example: Let \( X = (0, 1) \), \( F_n = (0, \frac{1}{n}] ; \ n \geq 2 \). Then \( \mathcal{J} = \{ F_n : n \geq 2 \} \) has f.i.p. But
\[ \cap F_n \in \mathcal{J} \neq \emptyset \]
Let \( F = \{ F_n : n \geq 2 \} \) be F.i.p. But \( \bigcap F \neq \emptyset \)

**Proposition 4.4:** A set \( K \subseteq X \) is compact if and only if every collection \( F \) of closed subsets of \( K \) with f.i.p. satisfies \( \bigcap F \neq \emptyset \).

**Proof:** See P21 in the book.

**Corollary 4.5:** Every compact metric space is complete.

**Corollary 4.6:** If \( X \) is compact, then every infinite set has a limit pt in \( X \).

**Proof:** Let \( S \) be an infinite subset of \( X \) and suppose \( S \) has no limit pts. Let \( \{ a_1, a_2, \ldots \} \) be a sequence of distinct pts in \( S \); then \( F_n = \{ a_1, a_2, \ldots \} \) also has no limit pts.

This implies \( F_n \) is closed.

Note: \( F = \{ F_n : n \geq 1 \} \) has f.i.p.

However, since \( a_1, a_2, \ldots \) are distinct,
\[
\bigcap_{n=1}^{\infty} F_n = \emptyset,
\]
contradicting proposition 4.5.

**Definition:** A metric space \((X,d)\) is sequentially compact if every sequence in \( X \) has a convergent subsequence.

**Lebesgue's Covering Lemma**

If \((X,d)\) is sequentially compact and \( G \) is an open cover of \( X \) then there is an \( \varepsilon > 0 \) st if \( x \) is in \( X \), there is an set \( G \subseteq G \) with \( B(x;\varepsilon) \subseteq G \).

**Proof:** We prove by contradiction. Suppose \( G \) is an open cover of \( X \) and no such \( \varepsilon > 0 \) can be found. In particular, for every \( n \in \mathbb{Z}^+ \) there is a pt \( x_n \in X \) such that \( B(x_n, \frac{1}{n}) \) is not contained in any set \( G \) in \( G \). Since \( X \) is sequentially compact, there is a pt \( x_0 \in X \) and a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) s.t. \( x_0 = \lim x_{n_k} \). Let \( G_0 \subseteq G \) st \( x_0 \in G_0 \). There \( \exists \varepsilon > 0 \) st \( B(x_0, \varepsilon) \subseteq G_0 \). Now let
\[ \{x_n \} \text{ s.t. } x_0 = \lim x_n \text{. Let } \epsilon > 0 \text{.} \]

Let \( x_0 \in G_0 \). There exists \( \epsilon > 0 \) s.t. \( B(x_0, \epsilon) \subset G_0 \). Now let \( N \) be such that \( d(x_0, x_n) < \frac{\epsilon}{2} \) for all \( n \geq N \).

Let \( N_0 = \max \{ N, \frac{1}{\epsilon} \} \). Let \( y \in B(x_0, \frac{1}{n_0}) \). Then

\[ d(x_0, y) \leq d(x_0, x_{n_0}) + d(x_{n_0}, y) < \frac{\epsilon}{2} + \frac{1}{n_0} < \epsilon. \]

That is, \( B(x_0, \frac{1}{n_0}) \subset B(x_0, \epsilon) \subset G_0 \), contradicting the choice of \( x_{n_0} \).

**Theorem 4.9.** Let \((X, \delta)\) be a metric space. Then the following are equivalent (TFAE):

(a) \( X \) is compact;

(b) Every infinite set in \( X \) has a limit pt. 

(c) \( X \) is sequentially compact

(d) \( X \) is complete and for every \( \epsilon > 0 \) there are a finite number of pts \( x_1, \ldots, x_k \) in \( X \) s.t.

\[ x \in \bigcup_{k=1}^{k} B(x_k, \epsilon). \]

(The property in (d) is called total boundedness)

**Pf:** See Page 22 in the book.

**Theorem 4.10 (Heine-Borel Theorem)**

A subset \( K \) of \( \mathbb{R}^n (n \geq 1) \) is compact iff \( K \) is closed and bounded.

**Pf:** Page 23 in book.