5. Continuity

**Defn 5.1:** Let \((X, d)\) and \((\mathbb{R}, \rho)\) be metric spaces and let \(f : X \to \mathbb{R}\) be a function. If \(a \in X\) and \(w \in \mathbb{R}\), then
\[
\lim_{x \to a} f(x) = w \text{ if for every } \varepsilon > 0 \text{ there is a } \\
\delta > 0 \text{ s.t. } \rho(f(x), w) < \varepsilon \text{ whenever } 0 < d(x, a) < \delta.
\]

The function \(f\) is continuous at the pt \(a\) if
\[
\lim_{x \to a} f(x) = f(a)
\]

If \(f\) is continuous at each pt of \(X\) then \(f\) is a continuous function from \(X\) to \(\mathbb{R}\).

**Proposition 5.2:** Let \(f : (X, d) \to (\mathbb{R}, \rho)\) be a function and \(a \in X, \alpha = f(a)\). TFAE

(a) \(f\) is continuous at \(a\);
(b) For every \(\varepsilon > 0\), \(f^{-1}(B(\alpha; \varepsilon))\) contains a ball with center at \(a\).
(c) \(\alpha = \lim f(x_n)\) whenever \(a = \lim x_n\).

**Pf:** we will only prove (a) \(\iff\) (b).

\(\Rightarrow\) For every \(\varepsilon > 0\), by (b), there \(\exists\) \(\delta > 0\) s.t. \(B(a, \delta) \subset f^{-1}(B(\alpha; \varepsilon))\). That is: \(f(B(a, \delta)) \subset B(\alpha; \varepsilon)\). This implies for all \(x \in X\) with \(0 < d(x, a) < \delta\), we have

\[f(x) \in B(\alpha, \varepsilon) \quad \text{ Hence} \]
for all $x \in X$ with $0 < d(x, a) < \delta$, we have

$$P(f(x), x) < \varepsilon.$$ Hence

$$f(a) = \lim_{x \to a} f(x).$$

2. $\implies$ Easy.

Proposition 3.3: Let $f : (X, d) \to (\mathbb{R}, p)$ be a function.

TFAE:
(a) $f$ is continuous;
(b) If $\Delta$ is open in $\mathbb{R}$ then $f^{-1}(\Delta)$ is open in $X$;
(c) If $\Gamma$ is closed in $\mathbb{R}$ then $f^{-1}(\Gamma)$ is closed in $X$.

Proof: (a) $\implies$ (b). Exercise.

(b) $\implies$ (c). If $\Gamma \subseteq \mathbb{R}$ is closed then let

$$\Delta = \mathbb{R} - \Gamma.$$ By (b), $f^{-1}(\Delta) = X - f^{-1}(\Gamma)$ is open $\implies f^{-1}(\Gamma)$ is closed.

(c) $\implies$ (a). Suppose there is a pt $x \in X$ at which $f$ is not continuous. Then there is an $\varepsilon > 0$ and a sequence $\{x_n\}$
s.t. $P(f(x_n), f(x)) \geq \varepsilon$ for every $n$ while

$$x = \lim_{n \to \infty} x_n.$$ Let $\Gamma = \mathbb{R} - B(f(x), \varepsilon)$;
then $\Gamma$ is closed and each $x_n$ is in $f^{-1}(\Gamma)$.
Since (by (c)) $f^{-1}(\Gamma)$ is closed we have

$$x \notin f^{-1}(\Gamma).$$
But this implies $P(f(x), f(x)) \geq \varepsilon > 0$. This is a contradiction.
This implies \( \lim_{n \to \infty} f_n(x) = 0 \) for every \( x \in X \), which implies \( \sup_{n \geq 1} \| f_n \| = 0 \) and is a contradiction.

**Proposition 5.4:** Let \( f \) and \( g \) be continuous functions from \( X \) into \( C \) and let \( \alpha, \beta \in C \). Then
\[
\alpha f + \beta g \text{ and } fg \text{ are both continuous.}
\]
Also, \( f/g \) is continuous provided \( g(x) \neq 0 \) for every \( x \in X \).

**Proposition 5.5:** Let \( f : X \to Y \) and \( g : Y \to Z \) be continuous functions. Then \( g \circ f \) is a continuous function from \( X \) into \( Z \).

**Defn:** A function \( f : (X, \mathcal{O}) \to (Y, \mathcal{P}) \) is uniformly continuous if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \), where \( \delta \) depends on \( \varepsilon \), such that \( \rho(f(x), f(y)) < \varepsilon \) whenever \( d(x, y) < \delta \). We say that \( f \) is a Lipschitz function if there is a constant \( M > 0 \) such that \( \rho(f(x), f(y)) \leq M d(x, y) \) for all \( x, y \in X \).

**Remark:** Lipschitz \( \Rightarrow \) uniformly continuous

**Example:**
(a) Let \( X = \mathbb{R} = (0, 1) \). Then \( f(x) = x^2 \) is continuous but not uniformly continuous, not Lipschitz.
(b) Let \( X = \mathbb{R} = (0, 1) \). Then \( f(x) = x^2 \) is uniformly continuous but not Lipschitz.

Let \( A \) be a non-empty subset of \( X \) and \( x \in X \); define the distance from \( x \) to \( A \), \( \delta(x, A) \), by
\[
\delta(x, A) = \inf_{a \in A} \{ d(x, a) \}.
\]
define the distance from $x$ to $A$, $d(x, A)$, by

$$d(x, A) = \inf \{ d(x, a) : a \in A \}.$$ 

**Proposition 5.7.** Let $A$ be a non-empty subset of $X$; then:

(a) $d(x, A) = d(x, A^-)$

(b) $d(x, A) = 0$ iff $x \in A$

(c) $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in A$

**Theorem 5.8:** Let $f: (X, d) \to (\mathbb{R}, \rho)$ be a continuous function.

(a) If $X$ is compact then $f(X)$ is a compact subset of $\mathbb{R}$;

(b) If $X$ is connected then $f(X)$ is a connected subset of $\mathbb{R}$.

Proof: We only prove (a) here. Let $(x_n)$ be a sequence in $\mathbb{R}$. Then there is, for each $n$, a pt $x_n \in X$ with $w_n = f(x_n)$.

Since $X$ is compact there is a pt $x \in X$ and a subsequence $(x_{n_k})$ s.t. $x = \lim x_{n_k}$. Write $w = f(x)$. By the continuity of $f$, we have $w = \lim w_{n_k} = \lim f(x_{n_k})$. Hence $\mathbb{R}$ is sequentially compact, and is thus compact.

**Corollary 5.9:** If $f: X \to \mathbb{R}$ is continuous and $K \subset X$ is either compact or connected in $X$, then $f(K)$ is compact or connected, respectively, in $\mathbb{R}$.

**Corollary 5.10:** If $f: X \to \mathbb{R}$ is continuous and $X$ is
Corollary 5.10: If \( f : X \rightarrow \mathbb{R} \) is continuous and \( X \) is connected then \( \text{fix}(f) \) is an interval.

Corollary 5.12: If \( f : X \rightarrow \mathbb{R} \) is continuous and \( K \subset X \) is compact then there are pts \( x_0 \) and \( y_0 \) in \( K \) with \( f(x_0) = \sup \{ f(x) : x \in K \} \) and \( f(y_0) = \inf \{ f(y) : y \in K \} \).

\[ f(x_0) = \sup \{ f(x) : x \in K \} \]
\[ f(y_0) = \inf \{ f(y) : y \in K \} \]

**Proof:** If \( \alpha = \sup \{ f(x) : x \in K \} \) then \( \alpha \) is in \( K \) as \( f(K) \) is compact.

Corollary 5.13: If \( K \subset X \) is compact and \( f : X \rightarrow \mathbb{R} \) is continuous then there \( \exists x_0, y_0 \in K \) with

\[ f(x_0) = \sup \{ f(x) : x \in K \} \]
\[ f(y_0) = \inf \{ f(x) : x \in K \} \]

**Proof:** Note \( |f(x)| \) is a continuous function from \( X \) to \( \mathbb{R} \).

Corollary 5.14: If \( K \) is non-empty compact subset of \( X \) and \( x \) is in \( X \) then \( \exists y \in K \) s.t. \( d(x, y) = d(x, K) \).

Theorem 5.15: Suppose \( f : X \rightarrow \mathbb{R} \) is continuous and \( P \) is unbounded continuous.
Theorem 5.15: Suppose \( f: X \to Y \) is continuous and \( X \) is compact; then \( f \) is uniformly continuous.

\[
\text{Pf.} \quad \text{Fix } \varepsilon > 0. \text{ Since } f \text{ is continuous, for every } x \in X, \\
\exists \delta_x > 0 \text{ s.t. } B(x, \delta_x) \subseteq f^{-1}(B(f(x), \frac{\varepsilon}{2})). \text{ Note} \\
\{ B(x, \frac{\delta_x}{2}) : x \in X \} \text{ is an open cover of } X. \\
\text{Thus } \exists \text{ finitely many } x_1, \ldots, x_N \text{ s.t.} \\
\{ B(x_j, \frac{\delta_{x_j}}{2}) \}_{j=1}^N \text{ covers } X. \text{ Let } 0 < \delta < \min\{ \frac{\delta_{x_j}}{2} : 1 \leq j \leq N \}. \\
\text{Then for any } x, y \in X \text{ with } |x - y| < \delta, \text{ we have there } \\
\exists \text{ some } j_0 \text{ s.t. } x \in B(x_{j_0}, \delta_{x_{j_0}}). \text{ Then } y \in B(x_{j_0}, \delta_{j_0}). \\
\text{By the choice of } \delta_{j_0}, \text{ we have } x, y \in f^{-1}(B(f(x_{j_0}), \frac{\varepsilon}{2})), \text{ i.e.} \\
\rho(f(y), f(x_{j_0})) < \frac{\varepsilon}{2} \\
\rho(f(x), f(x_{j_0})) < \frac{\varepsilon}{2} \\
\text{Thus } \rho(f(x), f(y)) < \varepsilon.
\]

Defn.: If \( A, B \) are non-empty subsets of \( X \) then define the distance from \( A \) to \( B \) by

\[
d(A, B) = \inf \{ d(a, b) : a \in A, b \in B \}
\]

Remark: \( \begin{align*}
\text{(1) } & d(A, B) = d(B, A) \\
\text{(2) If } B \text{ is a single set: } B = \{ x \} \text{ then} \\
d(A, B) = d(A, \{ x \}) = d(x, A). \\
\text{If } A = \{ y \}, \ B = \{ x \}, \text{ then} \\
d(A, B) = d(x, y)
\end{align*} \)
If $A = \{y\}$, $B = \{x\}$, then

$$d(A, B) = d(x, y)$$

3. If $A \cap B \neq \emptyset$, then $d(A, B) = 0$.

Example: Let $A = (0, 1)$, $B = (1, 2)$. Then $d(A, B) = 0$.

Thm 5.17: If $A, B$ are non-empty disjoint sets in $X$ with $B$ closed and $A$ compact then

$$d(A, B) > 0.$$  

Proof: Define $f: X \to \mathbb{R}$ by $f(x) = d(x, B)$. Since $A \cap B = \emptyset$ and $B$ is closed, we have, for $a \in A$, $a \notin B$. Thus $d(a, B) > 0$. But since $A$ is compact, there is a pt $a \in A$ s.t.

$$0 < f(a) = \inf \{f(x) : x \in A\} = d(A, B).$$

6. Uniform convergence

Let $X$ be a set and $(M, P)$ a metric space and suppose $f, f_1, f_2, \ldots$ are functions from $X$ to $M$.

The sequence $\{f_n\}$ converges uniformly to $f$ - written $f = u-\lim f_n$ - if for every $\varepsilon > 0$ there is an integer $N$ (depending on $\varepsilon$ only) s.t.

$$P(f(x), f_n(x)) < \varepsilon$$

for all $x \in X$, whenever $n \geq N$. Hence

$$\sup P(f(x), f_n(x)) : x \in X \leq \varepsilon$$

whenever $n \geq N$.

Theorem: If $A : \mathbb{R} \to \mathbb{R}^2$ is continuous
Thm 6.1 Suppose \( f_n : (\mathcal{X}, d) \to (\mathcal{Y}, \rho) \) is continuous
for each \( n \) and that \( f = \lim u \)-\( f_n \), then
\( f \) is continuous.

**Proof:** Fix \( x_0 \in \mathcal{X} \) and \( \epsilon > 0 \), we wish to find a \( \delta > 0 \)

such that \( \rho(f(x_0), f(x)) < \epsilon \) when \( d(x_0, x) < \delta \).

Since \( f = \lim u \)-\( f_n \), there \( \exists \) sufficiently large \( n \) such that
\( \rho(f(x), f_n(x)) < \frac{\epsilon}{3} \) for all \( x \in \mathcal{X} \).

Since \( f_n \) is continuous, there \( \exists \delta > 0 \) such that
\[ \rho(f_n(x_0), f_n(x)) < \frac{\epsilon}{3} \text{ when } d(x_0, x) < \delta. \]

Therefore, if \( d(x_0, x) < \delta \),
\[ \rho(f(x_0), f(x)) \leq \rho(f(x_0), f_n(x_0)) + \rho(f_n(x_0), f_n(x)) + \rho(f_n(x), f(x)) \]
\[ < \epsilon. \]

Let us consider the special case where \( \mathcal{Y} = \mathbb{C} \).

If \( u_n : \mathcal{X} \to \mathbb{C}, \) let \( f_n(x) = u_1(x) + \cdots + u_n(x) \).

We say \( f(x) = \sum_{n=1}^{\infty} u_n(x) \) iff \( f(x) = \lim f_n(x) \) for \( \forall x \in \mathcal{X} \).

The series \( \sum_{n=1}^{\infty} u_n \) is uniformly convergent to \( f \) iff
\[ f = \lim u \)-\( f_n \).

**Thm 6.2 (Weierstrass M-Test)** Let \( u_n : \mathcal{X} \to \mathbb{C} \) be

a function s.t. \( |u_n(x)| \leq M_n \) for \( \forall x \in \mathcal{X} \) and

Suppose \( \sum_{n=1}^{\infty} M_n < \infty \). Then
\[ \sum_{n=1}^{\infty} u_n \text{ is uniformly convergent.} \]

**Proof:** Page 29 in book.