5. Continuity

Defn 5.1: Let (X, d) and (Ω, ρ) be metric spaces and let $f: X \to \Omega$ be a function. If $a \in X$ and $w \in \Omega$, then $\lim_{x\to a} f(x) = w$ if for every $\varepsilon > 0$ there is a $\delta > 0$ s.t $\rho(f(x), w) < \varepsilon$ whenever $0 < d(x, a) < \delta$.

The function f is continuous at the pt a if

$$\lim_{x \to a} f(x) = f(a)$$

If f is continuous at each pt of X then f is a continuous function from X to Ω.

Proposition 5.2: Let $f: (X, d) \to (\Omega, \rho)$ be a function and $a \in X$, $\alpha = f(a)$. TFAE

(a) f is continuous at a;

(b) For every $\varepsilon > 0$, $f^{-1}(B(\alpha; \varepsilon))$ contains a ball with center at a.

(c) $\alpha = \lim f(x_n)$ whenever $a = \lim x_n$.

Pf: we will only prove (a) $\Leftrightarrow$ (b).

① "$\Leftarrow$" For every $\varepsilon > 0$, by (b), there $\exists$ a $\delta > 0$ s.t $B(a, \delta) \subset f^{-1}(B(\alpha; \varepsilon))$.

That is. $f(B(a, \delta)) \subset B(\alpha; \varepsilon)$. This implies for $\forall\, x \in X$ with $0 < d(x, a) < \delta$, we have

$\rho(f(x), \alpha) < \varepsilon$. Hence

for $\forall x \in X$ with $0 < d(x,a) < \delta$, we have
$\rho(f(x), \alpha) < \varepsilon$. Hence
$$f(a) = \lim_{x \to a} f(x).$$

② "$\Rightarrow$" Easy.

proposition 5.3: Let $f: (X,d) \to (\Omega, \rho)$ be a function. TFAE:

(a) f is continuous;

(b) If Δ is open in Ω then $f^{-1}(\Delta)$ is open in X

(c) If Γ is closed in Ω then $f^{-1}(\Gamma)$ is closed in X.

pf: (a) $\Rightarrow$ (b) Excercise.

(b) $\Rightarrow$ (c): If $\Gamma \subset \Omega$ is closed then let $\Delta = \Omega - \Gamma$. By (b), $f^{-1}(\Delta) = X - f^{-1}(\Gamma)$ is open $\Rightarrow f^{-1}(\Gamma)$ is closed.

(c) $\Rightarrow$ (a): Suppose there is a pt $x \in X$ at which f is NOT continuous. Then there is an $\varepsilon > 0$ and a sequence $\{x_n\}$ s.t $\rho(f(x_n), f(x)) \geq \varepsilon$ for every n while $x = \lim x_n$. Let $\Gamma = \Omega - B(f(x), \varepsilon)$; then Γ is closed and each x_n is in $f^{-1}(\Gamma)$. Since (by (c)) $f^{-1}(\Gamma)$ is closed we have
$$x \in f^{-1}(\Gamma).$$
But this implies $\rho(f(x), f(x)) \geq \varepsilon > 0$. This is a contradiction.

But this implies $|f(x_n), f(x)| \geq \varepsilon > 0$. This is a contradiction.

Proposition 5.4: Let f and g be continuous functions from X into $\mathbb{C}$ and let $\alpha, \beta \in \mathbb{C}$. Then $\alpha f + \beta g$ and fg are both continuous. Also, f/g is continuous provided $g(x) \neq 0$ for every $x \in X$.

Proposition 5.5: Let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Then $g \circ f$ is a continuous function from X into Z.

Defn: A function $f: (X, d) \to (\Omega, \rho)$ is uniformly continuous if for every $\varepsilon > 0$ there is a $\delta > 0$, where δ depends on ε, s.t $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. We say that f is a Lipschitz function if there is a constant $M > 0$ s.t $\rho(f(x), f(y)) \leq M d(x, y)$ for all $x, y \in X$.

Remark: Lipschitz $\Rightarrow$ uniformly continuous

Example: (a) Let $X = \Omega = \mathbb{R}$. Then $f(x) = x^2$ is continuous but not uniformly continuous, not Lipschitz.

(b) Let $X = \Omega = [0, 1]$. Then $f(x) = x^{\frac{1}{2}}$ is uniformly continuous but NOT Lipschitz.

Let A be a non-empty subset of X and $x \in X$; define the distance from x to A, $d(x, A)$, by

$$d(x, A) = \inf \{ d(x, a) : a \in A \}$$

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Proposition 5.7: Let A be a non-empty subset of X; then:

(a) $d(x,A) = d(x,A^-)$

(b) $d(x,A) = 0$ iff $x \in A^-$

(c) $|d(x,A) - d(y,A)| \le d(x,y)$ for all $x, y \in A$

Thm 5.8: Let $f : (X, d) \to (\Omega, \rho)$ be a continuous function

(a) If X is compact then $f(X)$ is a compact subset of Ω;

(b) If X is connected then $f(X)$ is a connected subset of Ω.

pf: We only prove (a) here. Let $\{w_n\}$ be a sequence in Ω. Then there is, for each n, a pt $x_n \in X$ with $w_n = f(x_n)$.

Since X is compact there is a pt $x \in X$ and a subsequence $\{x_{n_k}\}$ s.t $x = \lim x_{n_k}$. Write $w = f(x)$. By the continuity of f, we have $w = \lim w_{n_k} = \lim f(x_{n_k})$. Hence Ω is sequentially compact, and is thus compact.

Corollary 5.9: If $f : X \to \Omega$ is continuous and $K \subset X$ is either compact or connected in X then $f(K)$ is compact or connected, respectively, in Ω.

Corollary 5.10: If $f : X \to \mathbb{R}$ is continuous and X is

Corollary 5.10: If $f: X \to \mathbb{R}$ is continuous and X is connected then $f(X)$ is an interval.

Corollary 5.12: If $f: X \to \mathbb{R}$ is continuous and $K \subset X$ is compact then there are pts x_0 and y_0 in K with $f(x_0) = \sup\{f(x) : x \in K\}$ and

$$f(y_0) = \inf\{f(y) : y \in K\}.$$

Pf: If $\alpha = \sup\{f(x) : x \in K\}$ then α is in K as $f(K)$ is compact.

Corollary 5.13: If $K \subset X$ is compact and $f: X \to \mathbb{C}$ is continuous there there $\exists\ x_0, y_0 \in K$ with

$$|f(x_0)| = \sup\{|f(x)| : x \in K\}$$
$$|f(y_0)| = \inf\{|f(x)| : x \in K\}$$

Pf: Note $|f(x)|$ is a continuous function from X to $\mathbb{R}$.

Corollary 5.14: If K is non-empty compact subset of X and x is in X then $\exists\ y \in K$ s.t. $d(x, y) = d(x, K)$

Theorem 5.15: Suppose $f: X \to \Omega$ is continuous and X is compact, then f is uniformly continuous

Theorem 5.15: Suppose $f: X \to \cup L$ is continuous and X is compact; then f is uniformly continuous.

Pf: Fix $\varepsilon > 0$. Since f is continuous, for every $x \in X$, $\exists\, \delta_x > 0$ s.t $B(x, \delta_x) \in f^{-1}(B(f(x), \frac{\varepsilon}{2}))$. Note $\{B(x, \frac{\delta_x}{2}) : x \in X\}$ is an open cover of X.

Thus $\exists$ finitely many $x_1, \cdots, x_N$ s.t $\{B(x_j, \frac{\delta_{x_j}}{2})\}_{j=1}^{N}$ covers X. Let $0 < \delta < \min\{\frac{\delta_{x_j}}{2} : 1 \le j \le N\}$.

Then for any $x, y \in X$ with $|x-y| < \delta$. we have there $\exists$ some j_0 s.t $x \in B(x_{j_0}, \frac{\delta_{x_{j_0}}}{2})$. Then $y \in B(x_{j_0}, \delta_{j_0})$.

By the choice of δ_{j_0}, we have $x, y \in f^{-1}(B(f(x), \frac{\varepsilon}{2}))$, i.e

$$\rho(f(y), f(x_{j_0})) < \frac{\varepsilon}{2}$$

$$\rho(f(x), f(x_{j_0})) < \varepsilon/2$$

Thus $\rho(f(x), f(y)) < \varepsilon$.

Defn: If A, B are non-empty subsets of X then define the distance from A to B by

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

Remark: ① $d(A, B) = d(B, A)$

② If B is a single set: $B = \{x\}$ then

$$d(A, B) = d(A, \{x\}) = d(x, A).$$

If $A = \{y\}$, $B = \{x\}$, then

$$d(A, B) = d(x, y)$$

If $A=\{y\}$, $B=\{x\}$, then

$$d(A,B)=d(x,y)$$

③ If $A\cap B\neq\emptyset$, then $d(A,B)=0$.

Example: Let $A=(0,1)$, $B=(1,2)$. Then $d(A,B)=0$.

Thm 5.17: If A, B are non-empty disjoint sets in X with B closed and A compact then

$$d(A,B)>0.$$

Pf: Define $f: X\to\mathbb{R}$ by $f(x)=d(x,B)$. Since $A\cap B=\emptyset$ and B is closed, we have, for $a\in A$, $a\notin B^-$. Thus $d(a,B)>0$. But since A is compact, there is a pt $a\in A$ s.t

$$0<f(a)=\inf\{f(x): x\in A\}=d(A,B).$$

6. Uniform convergence

Let X be a set and (Ω,ρ) a metric space and Suppose $f, f_1, f_2, \cdots$ are functions from X to Ω. The sequence $\{f_n\}$ converges uniformly to f – written $f=u\text{-}\lim f_n$ – if for every $\varepsilon>0$ there is an integer N (depending on ε only) s.t

$$\rho(f(x), f_n(x))<\varepsilon \text{ for all } x\in X,$$

whenever $n\geq N$. Hence

$$\sup\{\rho(f(x), f_n(x)): x\in X\}\leq\varepsilon$$

whenever $n\geq N$.

Thm 6.1 Suppose $f_n: (X,d)\to(\Omega,\rho)$ is continuous

Thm 6.1 Suppose $f_n : (X, d) \to (\Omega, \rho)$ is continuous for each n and that $f = u\text{-}\lim f_n$; then f is continuous.

Pf: Fix $x_0 \in X$ and $\varepsilon > 0$. We wish to find a $\delta > 0$ s.t $\rho(f(x_0), f(x)) < \varepsilon$ when $d(x_0, x) < \delta$.

Since $f = u\text{-}\lim f_n$, there $\exists$ sufficiently large n_0 s.t $\rho(f(x), f_{n_0}(x)) < \varepsilon/3$ when for all $x \in X$.

Since f_{n_0} is continuous, there $\exists\ \delta > 0$ s.t

$$\rho(f_{n_0}(x_0), f_{n_0}(x)) < \frac{\varepsilon}{3} \text{ when } d(x_0, x) < \delta.$$

Therefore, if $d(x_0, x) < \delta$,

$$\rho(f(x_0), f(x)) \le \rho(f(x_0), f_{n_0}(x_0)) + \rho(f_{n_0}(x_0), f_{n_0}(x)) + \rho(f_{n_0}(x), f(x))$$
$$< \varepsilon.$$

Let us consider the special case where $\Omega = \mathbb{C}$.

If $u_n : X \to \mathbb{C}$, let $f_n(x) = u_1(x) + \cdots + u_n(x)$.

We say $f(x) = \sum_{n=1}^{\infty} u_n(x)$ iff $f(x) = \lim f_n(x)$ for $\forall x \in X$.

The series $\sum_{n=1}^{\infty} u_n$ is uniformly convergent to f iff $f = u\text{-}\lim f_n$.

Thm 6.2 (Weierstrass M-Test) Let $u_n : X \to \mathbb{C}$ be a function s.t $|u_n(x)| \le M_n$ for $\forall x \in X$ and suppose $\sum_{n=1}^{\infty} M_n < \infty$. Then

$$\sum_{n=1}^{\infty} u_n \text{ is uniformly Convergent.}$$

Pf: Page 29 in book.