1. power Series

Defn: If a_n is in $\mathbb{C}$ for every $n \geq 0$ then the Series $\sum_{n=0}^{\infty} a_n$ converges to z iff for every $\varepsilon > 0$ there $\exists$ an integer N such that

$$\left| \sum_{n=1}^{m} a_n - z \right| < \varepsilon.$$

whenever $m \geq N$.

The series $\sum a_n$ converges absolutely if $\sum |a_n|$ Converges

proposition 1.1: If $\sum a_n$ Converges absolutely then $\sum a_n$ converges.

Pf: Let $\varepsilon > 0$ and put $z_n = a_0 + a_1 + \cdots + a_n$.
Since $\sum |a_n|$ converges there $\exists N$ s.t $\sum_{n=N}^{\infty} |a_n| < \varepsilon$. Thus, if $m > k \geq N$,

$$|z_m - z_k| = \left| \sum_{n=k+1}^{m} a_n \right| \leq \sum_{n=k+1}^{m} |a_n| \leq \sum_{n=k+1}^{\infty} |a_n| < \varepsilon.$$

That is, $\{z_n\}$ is a Cauchy sequence and so there $\exists$ a $z \in \mathbb{C}$ with $z = \lim z_n$. Hence $\sum a_n = z$.

Defn: Let $\{a_n\}$ be a sequence in $\mathbb{R}$. Define

$$\liminf a_n = \lim_{n\to\infty} [\inf\{a_n, a_{n+1}, \cdots\}].$$

$$\limsup a_n = \lim_{n\to\infty} [\sup\{a_n, a_{n+1}, \cdots\}].$$

Sometimes, we also write

$$\liminf a_n = \underline{\lim}\, a_n$$

$$\limsup a_n = \overline{\lim}\, a_n.$$

proposition: If $\{a_n\}$ is a convergent sequence in $\mathbb{R}$ and proposition: If $\{a_n\}$ is a convergent sequence in $\mathbb{R}$ and $a = \lim a_n$. then $a = \liminf a_n = \limsup a_n$

proposition: $\liminf a_n \leq \limsup a_n$ for any sequence in $\mathbb{R}$.

A power series about a is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$.

Example: If $|z|<1$, then

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

Indeed, $$\sum_{n=0}^{m} z^n = 1 + \cdots + z^m = \frac{1-z^{m+1}}{1-z}$$

thus $$\lim_{m\to\infty} \sum_{n=0}^{m} z^n = \lim_{m\to\infty} \frac{1-z^{m+1}}{1-z} = \frac{1}{1-z}$$

Thm 1.3: For a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ define the number R, $0 \leq R \leq \infty$, by

$$\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}$$

then

(a) if $|z-a| < R$, the series converges absoultely;

(b) if $|z-a| > R$, the terms of the series become unbounded and so the series diverges;

(c) if $0 < r < R$ then the series converges uniformly on $\{z: |z-a| \leq r\}$.

Moreover, the number R is the only number having property (a) and (b).

Pf: We may suppose $a=0$. If $|z|<R$ there is an r with $|z|<r<R$. Thus, there is an integer N s.t $|a_n|^{1/n} < \frac{1}{r}$ for all $n \geq N$.

But then $|a_n| < \frac{1}{r^n}$ and so $|a_n z^n| < \left(\frac{|z|}{r}\right)^n$

Thus $\sum_{n=N}^{\infty} a_n z^n$ is dominated by $\sum \left(\frac{|z|}{r}\right)^n$. Since $\frac{|z|}{r} < 1$, we have $\sum \left(\frac{|z|}{r}\right)^n$ converges. Thus $\sum a_n z^n$ converges absolutely. This proves (a).

Now fix $r<R$ and choose $r<\rho<R$. As above, let N be such that $|a_n| < \frac{1}{\rho^n}$ for all $n \geq N$.

Then if $|z| \leq r$, $|a_n z^n| \leq \left(\frac{r}{\rho}\right)^n$ and $\left(\frac{r}{\rho}\right) < 1$.

Hence $\sum a_n z^n$ converges uniformly on $\{z : |z| \leq r\}$.

This proves (c).

Excercise: prove part (b).

Remark: The number R is called the radius of convergence of the power series.

proposition 1.4: If $\sum a_n (z-a)^n$ is a given power series with radius of convergence R, then

$$R = \lim \left|\frac{a_n}{a_{n+1}}\right|$$

if the limit exists.

Pf: Again assume that $a=0$ and let $\alpha = \lim \left|\frac{a_n}{a_{n+1}}\right|$.

Suppose that $|z|<r<\alpha$ and find N s.t

$\left|\frac{a_n}{a_{n+1}}\right| > r$ for all $n \geq N$. Let $B = |a_N| r^N$.

Then
$$|a_{N+1} r^{N+1}| = |a_{N+1}| r\, r^N < |a_N| r^N = B$$
$$|a_{N+2} r^{N+2}| = |a_{N+2}| r\, r^{N+1} < |a_{N+1}| r^{N+1} = B$$

Inductively, all $|a_n r^n| \leq B$ for $n \geq N$.

Indutively, all $|a_n r^n| \le B$ for $n \ge N$.

Then $|a_n z^n| = |a_n r^n| \frac{|z|^n}{r^n} \le B \frac{|z|^n}{r^n}$ for $n \ge N$.

Since $|z| < r$ we get $\sum |a_n z^n|$ converges. Since $r < \alpha$ was arbitrary this gives $\alpha \le R$.

On the other hand if $|z| > r > \alpha$, then $|a_n| < r|a_{n+1}|$ for all n large than some integer N. As before, we get $|a_n r^n| \ge B = |a_N r^N|$ for $n \ge N$.

This gives. $|a_n z^n| \ge B \frac{|z|^n}{r^n}$ which approaches ∞ as n does. Hence, $\sum a_n z^n$ diverses and so $R \le \alpha$. Thus $R = \alpha$.

Example: Consider $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Note $a_n = \frac{1}{n!}$. $R = \lim \left|\frac{a_n}{a_{n+1}}\right| = \infty$.

Thus this series has radius of convergence ∞.

We will designate the series by

$$e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

the exponential series or function.

Proposition 1.5: Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series and put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Then $\sum c_n$ is absolutely convergent with sum

$$\left(\sum a_n\right)\left(\sum b_n\right)$$

Proposition 1.6: Let $\sum a_n (z-a)^n$ and $\sum b_n (z-a)^n$ be power series with radius of convergence $\ge r > 0$.

power series with radius of convergence $\geq r > 0$.

put $$C_n = \sum_{k=0}^{n} a_k b_{n-k};$$

then both power series $\sum (a_n+b_n)(z-a)^n$ and $\sum C_n (z-a)^n$ have radius of convergence $r \geq 0$, and

$$\sum (a_n+b_n)(z-a)^n = \left[\sum a_n (z-a)^n + \sum b_n (z-a)^n\right]$$

$$\sum C_n (z-a)^n = \left[\sum a_n (z-a)^n\right]\left[\sum b_n (z-a)^n\right]$$

for $|z-a| < r$.

2. Analytic functions

Defn 2.1: If G is an open set in $\mathbb{C}$ and $f: G \to \mathbb{C}$ then f is differentiable at a pt $a \in G$ if

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists}.$$

Here $f'(a)$ will be called the derivative of f at a.

If f is differentiable at each pt of G, we say f is differentiable on G. Note if f is differentiable at each pt of G. we say f is differentiable on G. In this case, f' defines a function $f': G \to \mathbb{C}$. If f' is continuous then we say that f is continuously differentiable. If f' is differentiable then f is twice differentiable.

A differentiable function s.t each successive derivative is again differentiable is called infinitely differentiable.

proposition 2.2: If $f: G \to \mathbb{C}$ is differentiable at proposition 2.2: If $f: G \to \mathbb{C}$ is differentiable at $a \in G$, then f is continuous at a.

Defn 2.3: A function $f: G \to \mathbb{C}$ is analytic if f is continuously differentiable.

Chain Rule 2.4: Let f, g be analytic on G and Ω, respectively. and suppose $f(G) \subset \Omega$. Then $g \circ f$ is analytic on G and

$$(g \circ f)'(z) = g'(f(z)) f'(z).$$

for all $z \in G$.

In order to define the derivative, the function was assumed to be defined on an open set. If we say f is analytic on a set A and A is not open, we mean f is analytic on an open set containing A.

Example: Let $f(z) = \bar{z}$. Then f is NOT differentiable at a.

Indeed,
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\bar{h}}{h},$$

the limit does NOT exist.

proposition 2.5: Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$. Then

(a) For each $k \geq 1$ the series
$$\sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (z-a)^{n-k} \qquad (*)$$
has radius of convergence R.

(b) The function f is infinitely differentiable on $B(a; R)$ and furthermore, $f^{(k)}(z)$ is on $B(a;R)$ and furthermore, $f^{(k)}(z)$ is given by $(*)$ for all $k \geq 1$ and $|z-a| < R$.

(c) For $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Pf: Again assume that $a = 0$

(a) It suffices to prove for $k=1$. Recall

$$R^{-1} = \limsup |a_n|^{1/n}.$$

We wish to show

$$R^{-1} = \limsup |n a_n|^{\frac{1}{n-1}}$$

First we have

Claim: $\lim_{n\to\infty} n^{\frac{1}{n-1}} = 1$

Pf: Indeed, by L'Hopital's rule

$$\lim_{n\to\infty} \frac{\log n}{n-1} = 0$$

Thus $\lim_{n\to\infty} n^{\frac{1}{n-1}} = 1$

claim: $\limsup |a_n|^{\frac{1}{n-1}} = R$

Pf: Excercise.

Thus $$\limsup |n a_n|^{\frac{1}{n-1}} = R^{-1}$$

(b) we first prove for $k=1$.

For $|z|<R$ put $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$;

$S_n(z) = \sum_{k=0}^{n} a_k z^k$ and

$R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$.

Fix a pt $w \in B(0;R)$ and fix r with $|w| < r < R$. We wish to show that $f'(w)$ exists and equals to $g(w)$. To do this let

$|w| < r < R$. We wish to show that $f'(w)$ exists and equals to $g(w)$. To do this let $\delta > 0$ be arbitrary except for the restriction that $\overline{B}(w,\delta) \subset B(0,r)$. Let $z \in B(w,\delta)$.

Then

$$\frac{f(z)-f(w)}{z-w} - g(w) = \left[\frac{S_n(z)-S_n(w)}{z-w} - S_n'(w)\right] + \left[S_n'(w) - g(w)\right] + \left[\frac{R_n(z)-R_n(w)}{z-w}\right] \quad (**)$$

Now

$$\frac{R_n(z)-R_n(w)}{z-w} = \frac{1}{z-w}\sum_{k=n+1}^{\infty} a_k(z^k - w^k)$$
$$= \sum_{k=n+1} a_k\left(\frac{z^k-w^k}{z-w}\right)$$

Note

$$\frac{|z^k-w^k|}{|z-w|} = |z^{k-1}+z^{k-2}w+\dots+zw^{k-2}+w^{k-1}|$$
$$\leq kr^{k-1}$$

Hence $$\left|\frac{R_n(z)-R_n(w)}{z-w}\right| \leq \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}$$

Since $r<R$, $\sum_{k=1}^{\infty} |a_k| k r^{k-1}$ converges and so for any $\varepsilon > 0$ there is an integer N_1 such that for $n \geq N_1$

$$\left|\frac{R_n(z)-R_n(w)}{z-w}\right| < \frac{\varepsilon}{3}.$$

Also, $\lim S_n'(w) = g(w)$ so there $\exists$ an integer N_2 s.t

$$|S_n'(w) - g(w)| < \frac{\varepsilon}{3} \text{ whenever } n \geq N_2.$$

Let $n = \max\{N_1, N_2\}$. Then we can choose $\delta > 0$ s.t

$$\left|\frac{S_n(z)-S_n(w)}{z-w} - S_n'(w)\right| < \frac{\varepsilon}{3}$$

whenever $0 < |z-w| < \delta$. Putting all these inequalities together with $(**)$ we have that inequalities together with (**) we have that

$$\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| < \varepsilon$$

for $0<|z-w|<\delta$. That is, $f'(w)=g(w)$.
The general k case can be proved inductively.

(c) By a straightforward evaluation we get $f(0)=f^{(0)}(0)=a_0$. Using (a), we get $f^{(k)}(0)=k!\,a_k$ and this proves (c)

Corollary 2.9: If the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $R>0$ then $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ is analytic in $B(a;R)$.

Example: $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is analytic in $\mathbb{C}$.

Proposition 2.10: If G is open and connected and $f: G\to\mathbb{C}$ is differentiable with $f'(z)=0$ for all $z\in G$, then f is constant.

Pf: Fix $z_0\in G$. Let $w_0=f(z_0)$. put $A=\{z\in G: f(z)=w_0\}$. We will show that $A=G$ by proving that A is both open and closed in G. Let $z\in G$ be a limit pt of A. Then $\exists \{z_n\}\subset A$ s.t $z=\lim z_n$. Since $f(z_n)=w_0$ for each $n\geq 1$ and f is continuous we get $f(z)=w_0$, thus $z\in A$. Hence, A is closed in G. Now fix $a\in A$, and let $\varepsilon>0$ be s.t $B(a;\varepsilon)\subset G$. If $z\in B(a;\varepsilon)$, set $g(t)=f(tz+(1-t)a)$, $0\leq t\leq 1$. Then

$$\frac{g(t)-g(s)}{} = \frac{g(t)-g(s)}{} \cdot \frac{(t-s)z+(s-t)a}{}$$

$$\frac{g(t)-g(s)}{t-s} = \frac{g(t)-g(s)}{(t-s)z+(s-t)a} \cdot \frac{(t-s)z+(s-t)a}{t-s}$$

Thus, if we let $t \to s$ we get

$$\lim_{t\to s} \frac{g(t)-g(s)}{t-s} = f'(sz+(1-s)a)\cdot(z-a) = 0$$

That is, $g'(s)=0$ for $0 \le s \le 1$, implying g is a constant.

Hence, $f(z)=g(1)=g(0)=f(a)=w_0$. That is,

$B(a;\varepsilon) \subset A$ and A is also open.

Now differentiate $f(z)=e^z$. Recall

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

By Proposition 2.5,

$$f'(z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^n = f(z).$$

Thus $\frac{d}{dz} e^z = e^z.$

Proposition: $e^{a+b} = e^a \cdot e^b$

Pf: Define $g(z) = e^z e^{c-z}$.

Then $g'(z) = e^z e^{c-z} + e^z(-e^{c-z}) = 0.$

Thus $g(z)$ is a constant. Hence

$$g(z) = g(0) = e^c.$$

$$\Rightarrow \quad e^z e^{c-z} = e^c$$

Let $z=a$, $c=a+b$. Then $e^{a+b} = e^a \cdot e^b$.

Remark: In particular, we have $e^z e^{-z} = 1$.

This implies $e^z \neq 0$ for $\forall z \in \mathbb{C}$, and

Remark: In particular, we have ...

This implies $e^z \neq 0$ for $\forall z \in \mathbb{C}$, and

$$e^{-z} = \frac{1}{e^z}$$

proposition: $\exp \bar{z} = \overline{\exp z}$

pf: All the coefficients of the power series of e^z are all real.

Corollary: (1) $|e^z| = e^{\operatorname{Re} z} \quad \forall z \in \mathbb{C}$

pf: $|e^z|^2 = e^z \overline{e^z} = e^z e^{\bar{z}} = e^{z+\bar{z}} = e^{2\operatorname{Re} z}$

Thus $|e^z| = e^{\operatorname{Re} z}$.

(2) For $\forall \theta \in \mathbb{R}$, we have

$$|e^{i\theta}| = 1$$

<u>Defn</u>. For $z \in \mathbb{C}$, we define

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n-1)!}$$

Note they agree with real power series if $z \in \mathbb{R}$

Excercise: prove each series has radius of convergence $R = +\infty$.

proposition: $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$\frac{1}{2}(e^{iz} \quad -iz)$

Proposition: $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

Pf: Recall $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Then one can prove by manipulating power series.

Corollary: (1) $\cos^2 z + \sin^2 z = 1$ for $\forall z \in \mathbb{C}$

(2) $e^{iz} = \cos z + i \sin z$ for $\forall z \in \mathbb{C}$

Remark: Hence, if $z = \theta \in \mathbb{R}$, we have

$$e^{i\theta} = \operatorname{cis}\theta = \cos\theta + i\sin\theta$$

This implies

$$z = |z| e^{i\theta}, \text{ where } \theta = \arg z.$$

Since $e^{x+iy} = e^x e^{iy}$ we have

$$|e^z| = \exp(\operatorname{Re} z) \text{ and } \arg e^z = \operatorname{Im} z.$$

Defn: A function f is periodic with period c if $f(z+c) = f(z)$ for all $z \in \mathbb{C}$.

Q: Find the period of e^z.

A: If c is a period of e^z, then $e^z = e^{z+c} = e^z e^c$

Thus $e^c = 1$. Since $|e^c| = e^{\operatorname{Re} c} = 1$

$\Rightarrow \operatorname{Re} c = 0$ Thus $c = i\theta$ for some $\theta \in \mathbb{R}$.

But $1 = e^c = e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow$

$\theta = 2\pi k$ for some $k \in \mathbb{Z}$.

Hence the period $c = 2\pi k i$

Defn: If G is an open and connected subset of $\mathbb{C}$.

Defn: If G is an open and connected subset of $\mathbb{C}$. and $f: G \to \mathbb{C}$ is a continuous function s.t $z = \exp f(z)$ for $\forall z \in G$ then f is a branch of the logarithm.

Remark: We must have $0 \notin G$.

proposition: If $G \subset \mathbb{C}$ is open and connected and f is a branch of $\log z$, then $g: G \to \mathbb{C}$ is a branch of $\log z$ iff $g(z) = f(z) + 2\pi k i$, for some $k \in \mathbb{Z}$

Pf: "$\Leftarrow$" Easy

"$\Rightarrow$" Assume g is a branch of $\log z$ on G,

set $h(z) = \frac{1}{2\pi i}[f(z) - g(z)]$.

Then $e^{2\pi i h(z)} = e^{f(z) - g(z)} = \frac{z}{z} = 1$ for $z \in G$

Thus $h(z) \in \mathbb{Z}$ for $\forall z \in G \Rightarrow h(G) \subset \mathbb{Z}$

Since $h(z)$ is continuous and G is connected, $h(G)$ must be a single pt in $\mathbb{Z}$

Thus $\frac{1}{2\pi i}[f(z) - g(z)] = k$ for some $k \in \mathbb{Z}$.

Exercise: Let $G = \mathbb{C} - \{z \in \mathbb{R} : z \le 0\}$. Then G is connected and each $z \in G$ can be uniquely written as

$$z = |z| e^{i\theta} \text{ where } -\pi < \theta < \pi.$$

For θ in the range, define $f(re^{i\theta}) = \log r + i\theta$ prove f is continuous and thus is a branch of the logarithm on G. It is called the principal prove f is continuous and thus is a branch of the logarithm on G. It is called the principal branch of the logarithm.

proposition: Let G and Ω be open subsets of $\mathbb{C}$. suppose that $f: G \to \mathbb{C}$ and $g: \Omega \to \mathbb{C}$ are continuous functions s.t $f(G) \subset \Omega$ and $g(f(z)) = z$ for $\forall z \in G$. If g is differentiable and $g'(z) \neq 0$. then f is differentiable and

$$f'(z) = \frac{1}{g'(f(z))}.$$

Furthermore, if g is analytic, then f is analytic.

Pf: Fix $a \in G$ and let $h \in \mathbb{C}$ s.t $h \neq 0$ and $a+h \in G$. Note $a = g(f(a))$ and $a+h = g(f(a+h))$ implies $f(a) \neq f(a+h)$. Also

$$1 = \frac{g(f(a+h)) - g(f(a))}{h}$$

$$= \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \cdot \frac{f(a+h) - f(a)}{h}$$

We then let $h \to 0$. Since $\lim_{h\to 0} f(a+h) - f(a) = 0$, we have

$$\lim_{h\to 0} \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} = g'(f(a))$$

It is nonzero by assumption. Hence

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \frac{1}{g'(f(a))}$$

This means $f'(a) = \frac{1}{g'(f(a))}$.

This means $f'(a) = \overline{g'(f(a))}$.

If $g(z)$ is analytic, i.e. $g'(z)$ is continuous, then f' is continuous, and thus f is analytic.

Defn: Let G be a connected and open subset in $\mathbb{C}$. and $f: G \to \mathbb{C}$ a branch of the log. Fix $b \in \mathbb{C}$.

Define $g: G \to \mathbb{C}$

$$g(z) = \exp(bf(z)).$$

Then g is called a branch of z^b.

If we write $g(z) = z^b$ as a function, we will always understand that $z^b = \exp(b \log z)$ where $\log z$ is the principal branch of the logarithm. Note z^b is analytic since $\log z$ is.

Defn: A region is an open connected subset of the complex plane $\mathbb{C}$.

We next discuss the Cauchy-Riemann equation.

Let $f: G \to \mathbb{C}$ be analytic and let

$$\begin{aligned} u(x,y) &= \operatorname{Re} f(x+iy) \\ v(x,y) &= \operatorname{Im} f(x+iy) \end{aligned} \quad \text{for } x+iy \in \mathbb{R}$$

We evaluate $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ in two ways.

① First let $h \to 0$ through real value of h.

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+h+iy) - f(x+iy)}{h}$$

$$\frac{f(z+h)-f(z)}{h} = \frac{f(x+h+iy)-f(x+iy)}{h}$$
$$= \frac{u(x+h,y)-u(x,y)}{h} + i\,\frac{v(x+h,y)-v(x,y)}{h}$$

Letting $h \to 0$ gives
$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\,\frac{\partial v}{\partial x}(x,y) \qquad (3)$$

② Now let $h = it \to 0$ where $t \in \mathbb{R}$. Then
$$\frac{f(z+it)-f(z)}{it} = -i\,\frac{u(x,y+t)-u(x,y)}{t} + \frac{v(x,y+t)-v(x,y)}{t}$$

Thus
$$f'(z) = -i\,\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) \qquad (4)$$

Equating (3) and (4), we get
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (5)$$

The equation (5) is called Cauchy-Riemann equations.

Suppose u, v have continuous second partial derivatives. Differentiating the Cauchy-Riemann equations, we get
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Hence
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (6)$$

Defn: Any real-valued function u with continuous second derivatives satisfying (6) is said to be harmonic.

Let G be region in the plane and let u and v be functions defined on G with continuous partial derivatives. Furthermore, suppose that u and v satisfy the Cauchy-Riemann equations. If $f(z) = u(z) + iv(z)$ then f can

Riemann equations. If $f(z)=u(z)+iv(z)$ then f can be shown to be analytic in G. To see this, let $z=x+iy\in G$, and let $B(z;r)\subset G$. If $h=s+it\in B(0;r)$ then

$$u(x+s,y+t)-u(x,y)=[u(x+s,y+t)-u(x,y+t)]+[u(x,y+t)-u(x,y)]$$

Applying the M.V.T. we get for each $s+it\in B(0;r)$ $\exists$ s_1, t_1 s.t

$$\begin{cases} u(x+s,y+t)-u(x,y+t)=u_x(x+s,y+t)s \\ u(x,y+t)-u(x,y)=u_y(x,y+t_1)t \end{cases} \qquad (7)$$

Let $\varphi(s,t)=[u(x+s,y+t)-u(x,y)]-[u_x(x,y)s+u_y(x,y)t]$.

Equation (7) yields

$$\frac{\varphi(s,t)}{s+it}=\frac{s}{s+it}[u_x(x+s_1,y+t)-u_x(x,y)]+\frac{t}{s+it}[u_y(x,y+t_1)-u_y(x,y)]$$

But $|s|\le|s+it|$, $|t|\le|s+it|$, $|s_1|<|s|$, $|t_1|<t$ and the fact that u_x and u_y are continuous gives that

$$\lim_{s+it\to0}\frac{\varphi(s,t)}{s+it}=0. \qquad (8)$$

Hence

$$u(x+s,y+t)-u(x,y)=u_x(x,y)s+u_y(x,y)t+\varphi(s,t)$$

where φ satisfies (8). Similarly

$$v(x+s,y+t)-v(x,y)=v_x(x,y)s+v_y(x,y)t+\psi(s,t)$$

where ψ satisfies

$$\lim_{s+it\to0}\frac{\psi(s,t)}{s+it}=0 \qquad (9)$$

Using the fact that u, v satisfy the Cauchy-Riemann equations it is easy to see that

$$\frac{f(z+s+it)-f(z)}{s+it}=u_x(z)+iv_x(z)+\frac{\varphi(s,t)+i\psi(s,t)}{s+it}$$

By (8) and (9), f is differentiable and $f'(z)=u_x(z)+iv_x(z)$.

$s+it$

By (8) and (9), f is differentiable and $f'(z) = u_x(z) + i v_x(z)$. Since u_x and v_x are continuous, f' is continuous and f is analytic. Thus we get

Thm 2.29. Let u and v be real-valued functions defined on a region G and suppose that u and v have continuous partial derivatives. Then $f: G \to \mathbb{C}$ defined by $f(z) = u(z) + i v(z)$ is analytic iff u and v satisfy the Cauchy-Riemann equations.

A natural question arises: Suppose G is a region in $\mathbb{C}$ and $u: G \to \mathbb{R}$ is harmonic. Does there exist a harmonic function $v: G \to \mathbb{R}$ s.t. $f = u + iv$ is analytic in G? If such a function v exists it is called a harmonic conjugate of u. If exists, the harmonic conjugate is NOT unique. But two harmonic conjugates of a harmonic function differ by a constant. Indeed, if v_1 and v_2 are two harmonic conjugates of u then

$i(v_1 - v_2) = (u + iv_1) - (u + iv_2)$ is analytic on G, and only takes purely imaginary values. Thus it must be a constant (See Exercise 14).

Exercise: Show $u(z) = \log|z|$ is harmonic on $G = \mathbb{C} - \{0\}$.

Then show it does not have a conjugate on G.

However, there are some regions for which every harmonic function has a conjugate.

Thm 2.30. Let G be either the whole plane $\mathbb{C}$ or some open disk.

Thm 2.30. Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u: G \to \mathbb{R}$ is a harmonic function then u has a harmonic conjugate.

Pf: Let $G = B(0;R)$, $0 < R \le \infty$, and let $u: G \to \mathbb{R}$ be a harmonic function. We will find a harmonic function v s.t u and v satisfy the Cauchy-Riemann equations. So define

$$v(x,y) = \int_0^y u_x(x,t)\,dt + \varphi(x) \qquad (10)$$

Here $\varphi(x)$ is to be determined s.t $v_x = -u_y$.

Note we already have $v_y(x,y) = u_x(x,y)$.

Using Leibniz's rule, we differentiate both sides with respect to x, we get

$$\begin{aligned} v_x(x,y) &= \int_0^y u_{xx}(x,t)\,dt + \varphi'(x) \\ &= -\int_0^y u_{yy}(x,t)\,dt + \varphi'(x) \\ &= -u_y(x,y) + u_y(x,0) + \varphi'(x) \end{aligned}$$

So we must have $\varphi'(x) = -u_y(x,0)$. Then it is easy to check that u and

$$v(x,y) = \int_0^y u_x(x,t) - \int_0^x u_y(s,0)\,ds$$

do satisfy the Cauchy-Riemann equations.

Q: Where was the fact that G is a disk or $\mathbb{C}$ used?