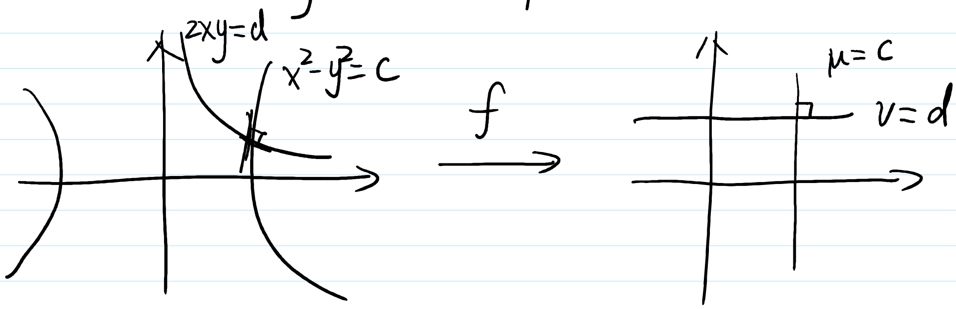
3. Analytic functions as mappings. Möbius transformations

We start with some examples:

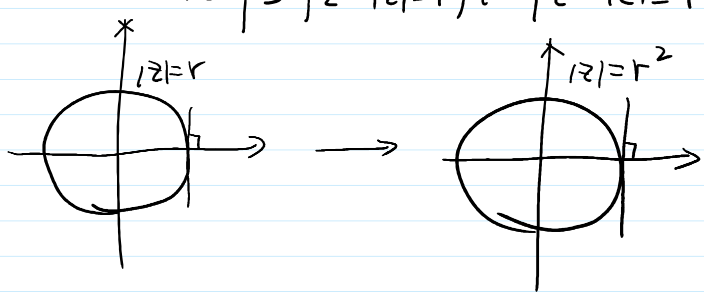
Example: Let $f(z)=z^2$. If we write $z=x+iy$ and $f(z)=u+iv$, then the hyperbolas $x^2-y^2=c$ and $2xy=d$ are mapped by f into the straight lines $u=c$, $v=d$. Note



Example: Let $f(z)=z^2$. Then

$f(z)$ maps x-axis to x-axis

maps $\{z: |z|=r\}$ to $\{z: |z|=r^2\}$.



Defn: A path is a region $G \subset \mathbb{C}$ is continuous function $\gamma: [a,b] \to G$ for some interval $[a,b]$ in $\mathbb{R}$. If $\gamma'(t)$ exists for each $t \in [a,b]$ and $\gamma': [a,b] \to \mathbb{C}$ is continuous then γ is a smooth path. Also γ is piecewise smooth if there is a partition of $[a,b]$, $a=t_0<t_1<\cdots<t_n=b$ s.t. γ is smooth on each

piecewise smooth if there is a partition of $[a,b]$,
$a=t_0<t_1<\cdots<t_n=b$ s.t γ is smooth on each
subinterval $[t_{j-1},t_j]$, $1\le j\le n$.

Remark: To say that a function $\gamma:[a,b]\to\mathbb{C}$ has a derivative $\gamma'(t)$ for each pt $t\in[a,b]$ means that

$$\lim_{h\to 0}\frac{\gamma(t+h)-\gamma(t)}{h}=\gamma'(t).$$

exists for $a<t<b$ and that the one-side limit exists for $t=a$ and $t=b$, respectively. This is equivalent to saying that $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ have a derivative.

Suppose $\gamma:[a,b]\to G$ is a smooth path and that for some t_0 in (a,b), $\gamma'(t_0)\ne 0$. Then γ has a tangent line at the pt $z_0=\gamma(t_0)$. This line goes through z_0 in the direction of $\gamma'(t_0)$
Let γ_1 and γ_2 be two smooth paths with $\gamma_1(t_1)=\gamma_2(t_2)$ and $\gamma_1'(t_1)$, $\gamma_2'(t_2)\ne 0$.

Defn: Define the angle between the paths γ_1 and γ_2 at z_0 to be

$$\arg\gamma_2'(z)-\arg\gamma_1'(z)$$

Suppose γ is a smooth path in G and $f:G\to\mathbb{C}$ is analytic.
Then $\sigma=f\circ\gamma$ is also a smooth path and

$$\sigma'(t)=f'(\gamma(t))\cdot\gamma'(t)$$

Let $z_0=\gamma(t_0)$, and suppose that $\gamma'(t_0)\ne 0$ and $f'(z_0)\ne 0$; then $\sigma'(t_0)\ne 0$ and

$$\arg\sigma'(t_0)-\arg\gamma'(t_0)=\arg f'(z_0)$$

$$\arg \sigma'(t_0) - \arg \gamma'(t_0) = \arg f'(z_0)$$

Now let γ_1 and γ_2 be smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$. and $\gamma_1'(t_1) \neq 0 \neq \gamma_2'(t_2)$; let $\sigma_1 = f\circ\gamma_1$, $\sigma_2 = f\circ\gamma_2$. Then

$$\arg \gamma_2'(t_2) - \arg \gamma_1'(t_1) = \arg \sigma_2'(t_2) - \arg \sigma_1'(t_1)$$

In this case, we say f preserves the angle at z_0 between the curves. The above argument yields

Thm 3.4: If $f: G \to \mathbb{C}$ is analytic then f preserves angles at each z_0 of G where $f'(z_0) \neq 0$.

A function $f: G \to \mathbb{C}$ which has the angle preserving property and also has $\lim\limits_{z\to a} \dfrac{|f(z)-f(a)|}{|z-a|}$ existing is called a conformal map.

Corollary: If f is analytic in G and $f'(z) \neq 0$ for $\forall z \in G$, then f is conformal.

The converse of this statement is also true.

<u>Defn</u>: A mapping of the form $S(z) = \dfrac{az+b}{cz+d}$ is called a linear fractional tranformation. If a, b, c and d also satisfy $ad-bc \neq 0$ then $S(z)$ is called a Möbius transformation.

Remark: The set of Möbius maps form a group under composition. Indeed, if $S = \dfrac{az+b}{cz+d}$ is a Möbius transformation, then $S^{-1} = \dfrac{dz-b}{}$ satisfies compo... Indeed, if $S = \frac{\;\;}{cz+d}$ ...

transformation, then $S^{-1} = \frac{dz-b}{-cz+a}$ satisfies $S(S^{-1}(z)) = z$, $S^{-1}(S(z)) = z$, i.e. S^{-1} is the inverse mapping of S. If S, T are both Möbius transformations, then $S \circ T$ is also. (Exercise)

Unless otherwise stated, the only linear fractional transformations we will consider are Möbius transformations

We will also consider S as defined on $\mathbb{C}_\infty$ with $S(\infty) = \frac{a}{c}$ and $S(-\frac{d}{c}) = \infty$ if $c \neq 0$; Note we have $S(\infty) = \infty$ when $c = 0$.

Remark: $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is one-to-one and onto.

Defn: We call

$S(z) = z + a$ a translation

$S(z) = az$, $a > 0$ a dilation

$S(z) = e^{i\theta} z$ a rotation

$S(z) = \frac{1}{z}$ the inversion

proposition 3.6: Every Möbius transformation is the composition of translations, dilations and the inversion

pf: ① If $c = 0$, then $S(z) = \frac{a}{d} z + \frac{b}{d}$. Hence

$$S = S_1 \circ S_2$$

where $S_1 = \frac{a}{d} z$, $S_2 = z + \frac{b}{d}$

② If $c \neq 0$, we put $S_1(z) = z + \frac{d}{c}$, $S_2(z) = \frac{1}{z}$

$(bc-ad)$ ... $S_4(z) = z + \frac{a}{c}$. Then

② If $c \neq 0$, we put $S_1(z) = z + \frac{d}{c}$, $S_2(z) = \frac{1}{z}$

$$S_3(z) = \frac{(bc-ad)}{c^2} z, \quad S_4(z) = z + \frac{a}{c} \quad \text{Then}$$

$$S = S_4 \circ S_3 \circ S_2 \circ S_1.$$

Remark: A Möbius transformation can have at most two fixed pts unless $S(z) = z$. Indeed,

$$\frac{az+b}{cz+d} = z \iff$$

$$cz^2 + (d-a)z - b = 0$$

It has at most two solutions if $c \neq 0$ or $d-a \neq 0$. or $b \neq 0$

Now let S be a Möbius transformation and let a, b, c be distinct pts in $\mathbb{C}_\infty$ with $\alpha = S(a)$, $\beta = S(b)$, $\gamma = S(c)$. Suppose that T is another map with this property. Then $T^{-1} \circ S$ has a, b and c as fixed pts and therefore, $T^{-1} \circ S = I$, where I is the identity. That is, $S = T$. Hence, a Möbius map is uniquely determined by its action on any three given pts in $\mathbb{C}_\infty$.

Let z_2, z_3, z_4 be pts in $\mathbb{C}_\infty$. Define $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$S(z) = \left(\frac{z-z_3}{z-z_4}\right) \Big/ \left(\frac{z_2-z_3}{z_2-z_4}\right) \quad \text{if } z_2, z_3, z_4 \in \mathbb{C}$$

$$S(z) = \frac{z-z_3}{z-z_4} \quad \text{if } z_2 = \infty$$

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$$S(z) = \frac{z - z_3}{z_2 - z_3} \quad \text{if } z_4 = \infty$$

In any case $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$ and S is only transformation having this property.

Defn 3.7: If $z_1 \in \mathbb{C}_\infty$ then (z_1, z_2, z_3, z_4), called the cross ratio of z_1, z_2, z_3 and z_4, is the image of z_1 under the unique Möbius transformation S with $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$.

Example: $(z_2, z_2, z_3, z_4) = 1$ and $(z, 1, 0, \infty) = z$. Also, if M is any Möbius map and w_2, w_3, w_4 are the pts s.t $Mw_2 = 1$, $Mw_3 = 0$, $Mw_4 = \infty$ then $Mz = (z, w_2, w_3, w_4)$.

Proposition 3.8: If z_2, z_3, z_4 are distinct pts and T is any Möbius transformation then

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$$

for any pt z_1.

Pf: Let $S(z) = (z, z_2, z_3, z_4)$, then S is a Möbius map. If $M = ST^{-1}$ then $M(Tz_2) = 1$, $M(Tz_3) = 0$, $M(Tz_4) = \infty$, hence,

$$ST^{-1}z = (z, Tz_2, Tz_3, Tz_4) \text{ for all } z$$

in $\mathbb{C}_\infty$. In particular, if $z = Tz_1$, the desired result follows.

Proposition 3.9: If z_2, z_3, z_4 are distinct pts in $\mathbb{C}_\infty$ and w_2, w_3, w_4 are also distinct pts of $\mathbb{C}_\infty$, then proposition 3.9: If z_2, z_3, z_4 are distinct pts in $\mathbb{C}_\infty$ and w_2, w_3, w_4 are also distinct pts of $\mathbb{C}_\infty$, then there is one and only one Möbius transformation S s.t $Sz_2 = w_2$, $Sz_3 = w_3$, $Sz_4 = w_4$.

Pf: Let $Tz = (z_1, z_2, z_3, z_4)$, $Mz = (z, w_2, w_3, w_4)$ and $S = M^{-1}T$. Clearly S has the desired property. If R is another Möbius map with $Rz_j = w_j$ for all $2 \le j \le 4$. Then $R^{-1} \circ S$ has three fixed pts z_2, z_3, and z_4. Hence $R^{-1} \circ S = I$ or $S = R$.

Remark: (1) A circle in $\mathbb{C}_\infty$ means either a circle in $\mathbb{C}$ or a line in $\mathbb{C}$

(2) Hence, three pts in $\mathbb{C}_\infty$ determines a circle in $\mathbb{C}_\infty$.

proposition 3.10: Let z_1, z_2, z_3, z_4 be four distinct pts in $\mathbb{C}_\infty$. Then $(z_1, z_2, z_3, z_4) \in \mathbb{R}$ iff all four pts lie on a circle.

Pf: We first prove the following Lemma:

Lemma: Let $Sz = \dfrac{az+b}{cz+d}$. Then $S^{-1}(\mathbb{R}_\infty)$ is a circle.

Pf: If $z = x \in \mathbb{R}$ and $w = S^{-1}(x) \neq \infty$, then $x = Sw$ implies $S(w) = \overline{S(w)}$. That is,

$x = Sw$ implies $S(w) = \overline{S(w)}$. That is,

$$\frac{aw+b}{cw+d} = \frac{\bar a \bar w + \bar b}{\bar c \bar w + \bar d}$$

Cross multiplying $\Rightarrow$

$$(a\bar c - \bar a c)|w|^2 + (a\bar d - \bar b c)w + (b\bar c - d\bar a)\bar w + (b\bar d - \bar b d) = 0 \quad (1)$$

If $a\bar c - \bar a c = 0$, putting $\alpha = z(a\bar d - \bar b c)$, $\beta = i(b\bar d - \bar b d) \in \mathbb{R}$.

and multiplying (1) by i gives

$$0 = \mathrm{Im}(\alpha w) - \beta = \mathrm{Im}(\alpha w - \beta) \quad (2)$$

This is a line in $\mathbb{C}$ and thus a circle in $\mathbb{C}_\infty$

If $a\bar c - \bar a c \neq 0$, then (1) $\Rightarrow$

$$|w|^2 + \bar\gamma w + \gamma \bar w - \delta = 0$$

for some $\gamma \in \mathbb{C}$, $\delta \in \mathbb{R}$. Hence

$$|w + \gamma| = (|\gamma|^2 + \delta)^{\frac{1}{2}} \quad (3)$$

This is a circle in $\mathbb{C}_\infty$ as well.

Then we have

Lemma: Let $S(z) = (z, z_2, z_3, z_4)$. Then $S^{-1}(\mathbb{R}_\infty) = \Gamma$, where Γ is the circle determined by z_2, z_3, z_4.

Moreover, $S(\Gamma) = \mathbb{R}_\infty$.

Pf: By the previous Lemma, $S^{-1}(\mathbb{R}_\infty)$ is a circle Γ in $\mathbb{C}^\infty$

On the other hand, $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$.

Thus $z_2, z_3, z_4 \in \Gamma$. Thus Γ is the circle determined by z_2, z_3, z_4. On the other hand, as $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is one-to-one and onto, thus $S(\Gamma) = \mathbb{R}_\infty$.

Thm 3.14: A Möbius transformation takes circles in $\mathbb{C}_\infty$ onto circles in $\mathbb{C}_\infty$

Pf: Let Γ be any circle in $\mathbb{C}_\infty$ and let S be any Möbius transformation. Let $z_2, z_3, z_4 \in \Gamma$ and put $w_j = Sz_j$ for $j = 2, 3, 4$. Then w_2, w_3, w_4 determine a circle Γ'. We claim that $S(\Gamma) = \Gamma'$. In fact, for $\forall z \in \mathbb{C}_\infty$, by proposition 3.8,

$$(z, z_2, z_3, z_4) = (Sz, w_2, w_3, w_4) \quad (4)$$

By proposition 3.10, if z is on Γ then both sides of (4) are real. But this says that $Sz \in \Gamma'$.

Let $T(z) = (z, z_2, z_3, z_4)$ and $M(w) = (w, w_2, w_3, w_4)$. Then $S(z) = M^{-1} \circ T(z)$. Hence $S(\Gamma) = M^{-1}(\mathbb{R}_\infty)$ is a circle in $\mathbb{C}_\infty$; this circle must be Γ'. Thus $S(\Gamma) = \Gamma'$.

Remark: Let Γ and Γ' be two circles in $\mathbb{C}_\infty$. Then $\exists$ a Möbius map T s.t. $T(\Gamma) = \Gamma'$. Indeed, Let $z_2, z_3, z_4 \in \Gamma$; $w_2, w_3, w_4 \in \Gamma'$. Put $Rz = (z, z_2, z_3, z_4)$. $Sz = (z, w_2, w_3, w_4)$. Then $T = S^{-1} \circ R$ maps Γ onto Γ'. This is because $Tz_j = w_j$, $2 \le j \le 4$. Then it follows from Thm 3.14 $T(\Gamma) = \Gamma'$. We summarize this as.

proposition 3.16: For any given circles Γ and Γ' in $\mathbb{C}_\infty$, there is a Möbius transformation T s.t. $T(\Gamma) = \Gamma'$.

there is a Möbius transformation T s.t $T(\Gamma)=\Gamma'$. Furthermore we can specify that T take any three pts on Γ onto any three pts of Γ'. If we do specify Tz_j for $j=2,3,4$ (distinct z_j in Γ), then T is unique.

Now we know a Möbius map takes circles to circles, the next question is; what happens to the inside and the outside of the circles? To understand this, we introduce:

<u>Defn</u>: Let Γ be a circle in $\mathbb{C}_\infty$. Then the points z, z^* in $\mathbb{C}_\infty$ are said to be symmetric with respect to Γ if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} \quad (4)$$

for some distinct pts $z_2, z_3, z_4 \in \Gamma$.

Proposition: Let Γ be a circle in $\mathbb{C}_\infty$. and $z, z^* \in \Gamma$. Then

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} \text{ for some}$$

distinct $z_2, z_3, z_4 \in \Gamma$. $\Longleftrightarrow$

$$(z^*, w_2, w_3, w_4) = \overline{(z, w_2, w_3, w_4)} \text{ for}$$

every choice of distinct pts $w_2, w_3, w_4 \in \Gamma$.

$(z^*, w_2, w_3, w_4) = \overline{(z, w_2, w_3, w_4)}$ for

every choice of distinct pts $w_2, w_3, w_4 \in \Gamma$.

Pf: Exercise. (Hint: First conside the case when $\Gamma = \mathbb{R}_\infty$).

Remark: By proposition 3.10, z is symmetric to itself with respect to Γ iff $z \in \Gamma$.

Let's investigate what it means for z and z^* to be symmetric. We have two cases:

① If Γ is a straight line, by the proposition above, we can choose z_2, z_3, z_4 as any distinct pts on Γ. We now fix $z_4 = \infty$. Then (4) becomes

$$\frac{z^* - z_3}{z_2 - z_3} = \frac{\bar{z} - \bar{z}_3}{\bar{z}_2 - \bar{z}_3}$$

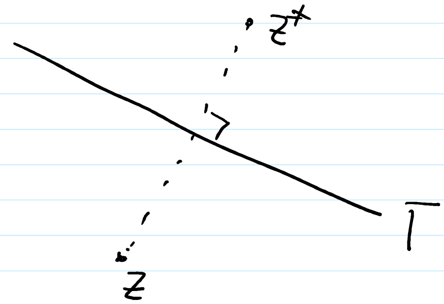
$$\Rightarrow \quad |z^* - z_3| = |z - z_3|$$

Since z_3 can be arbitrary pt on Γ, we have z and z^* are equidistant from each pt on Γ. Also

$$\operatorname{Im} \frac{z^* - z_3}{z_2 - z_3} = \operatorname{Im} \frac{\bar{z} - \bar{z}_3}{\bar{z}_2 - \bar{z}_3}$$

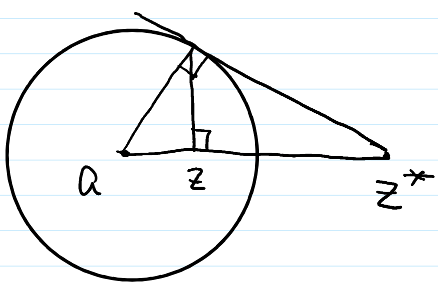
$$= - \operatorname{Im} \frac{z - z_3}{z_2 - z_3}$$

Thus z and z^* lies in two sides of Γ (unless $z \in \Gamma$)



② If $\Gamma = \{z : |z-a| = R\}$ is a circle. let z_2, z_3, z_4.

we have in this case:



$$
\begin{aligned}
(z^*, z_2, z_3, z_4) &= \overline{(z, z_2, z_3, z_4)} \\
&= \overline{(z-a, z_2-a, z_3-a, z_4-a)} \\
&= (\bar{z}-\bar{a}, \bar{z}_2-\bar{a}, \bar{z}_3-\bar{a}, \bar{z}_4-\bar{a}) \\
&= \left(\bar{z}-\bar{a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}, \frac{R^2}{z_4-a}\right) \\
&= \left(\frac{R^2}{\bar{z}-\bar{a}}, z_2-a, z_3-a, z_4-a\right) \\
&= \left(\frac{R^2}{\bar{z}-\bar{a}}+a, z_2, z_3, z_4\right)
\end{aligned}
$$

Hence $z^* = a + \frac{R^2}{\bar{z}-\bar{a}}$ or $(z^*-a)(\bar{z}-\bar{a}) = R^2$. $\Rightarrow$

$$\frac{z^*-a}{z-a} = \frac{R^2}{|z-a|^2} > 0.$$

so that z^* lies on the ray $\{a+t(z-a) : 0 < t < \infty\}$ from a through z.

Symmetry principle 3.19: If a Möbius transformation T takes a circle Γ_1 onto the circle Γ_2 then any pair of pts symmetric with respect to Γ_1 are mapped by T onto a pair of pts symmetric with respect to Γ_2.

of pts symmetric with respect to Γ_2.

Pf: Let $z_2, z_3, z_4 \in \Gamma_1$; it follows that if z and z^* are symmetric with respect to Γ_1 then by using proposition 3.8

$$(Tz^*, Tz_2, Tz_3, Tz_4) = (z^*, z_2, z_3, z_4)$$
$$= \overline{(z, z_2, z_3, z_4)}$$
$$= \overline{(Tz, Tz_2, Tz_3, Tz_4)}$$

Hence Tz^* and Tz are symmetric with respect to Γ_2.