1. Riemann-Stieltjes integral

The main purpose of this section is to define the integral of a function along a path in $\mathbb{C}$.

Defn 1.1: A function $\gamma: [a,b] \to \mathbb{C}$, for $[a,b] \subset \mathbb{R}$, is of bounded variation if there $\exists$ a constant $M > 0$ s.t for any partition $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a,b]$

$$v(\gamma; P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \le M$$

The total variation of γ, $V(\gamma)$, is defined by

$$V(\gamma) = \sup\{v(\gamma; P) : P \text{ a partition of } [a,b]\}.$$

Clearly $V(\gamma) \le M < \infty$.

Remark: γ is of bounded variation iff $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ are of bounded variation.

Example: If γ is real valued and is non-decreasing then γ is bounded variation and

$$V(\gamma) = \gamma(b) - \gamma(a).$$

proposition 1.2: Let $\gamma: [a,b] \to \mathbb{C}$ be of bounded variation. Then variation. Then

(a) If P and Q are partitions of $[a,b]$ and $P \subset Q$ then $v(\gamma;P) \le v(\gamma;Q)$.

(b) If $\sigma:[a,b]\to\mathbb{C}$ is also of bounded variation and $\alpha,\beta\in\mathbb{C}$ then $\alpha\gamma+\beta\sigma$ is of bounded variation and

$$V(\alpha\gamma+\beta\sigma) \le |\alpha|\,V(\gamma)+|\beta|\,V(\sigma).$$

Pf: Excercise.

Proposition 1.3: If $\gamma:[a,b]\to\mathbb{C}$ is piecewise smooth then γ is of bounded variation and

$$V(\gamma)=\int_a^b|\gamma'(t)|\,dt$$

Pf: Assume that γ is smooth (the complete proof is easily deduced from this). Recall that when we say that γ is smooth this implies γ' is continuous.

Let $P=\{a=t_0<t_1<\cdots<t_m=b\}$. Then, by definition,

$$\begin{aligned} v(\gamma;P) &= \sum_{k=1}^{m}|\gamma(t_k)-\gamma(t_{k-1})| \\ &= \sum_{k=1}^{m}\left|\int_{t_{k-1}}^{t_k}\gamma'(t)\,dt\right| \\ &\le \sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}|\gamma'(t)|\,dt \\ &= \int^{b}|\gamma'(t)|\,dt \end{aligned}$$

t_{k-1}

$$= \int_a^b |\gamma'(t)|\, dt$$

Hence $V(\gamma) \le \int_a^b |\gamma'(t)|$, so that γ is of bounded variation.

Since γ' is continuous it is uniformly continuous ; so if $\varepsilon > 0$ is given we can chose $\delta_1 > 0$ s.t $|s-t| < \delta_1$ implies that $|\gamma'(s) - \gamma'(t)| < \varepsilon$. Also, we may choose $\delta_2 > 0$ s.t if $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ and

$$\|P\| = \max\{(t_k - t_{k-1}) : 1 \le k \le m\} < \delta_2 \text{ then}$$

$$\left| \int_a^b |\gamma'(t)|\, dt - \sum_{k=1}^{m} |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \varepsilon$$

where τ_k is any point in $[t_{k-1}, t_k]$. Hence

$$\begin{aligned} \int_a^b |\gamma'(t)|\, dt &\le \varepsilon + \sum_{k=1}^{m} |\gamma'(\tau_k)|(t_k - t_{k-1}) \\ &\le \varepsilon + \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k)\, dt \right| \\ &\le \varepsilon + \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t)]\, dt \right| \\ &\quad + \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} \gamma'(t)\, dt \right| \end{aligned}$$

If $\|P\| < \delta = \min(\delta_1, \delta_2)$ then $|\gamma'(\tau_k) - \gamma'(t)| < \varepsilon$ for $t \in [t_{k-1}, t_k]$ and

$$\begin{aligned} \int_a^b |\gamma'(t)|\, dt &\le \varepsilon + \varepsilon(b-a) + \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \varepsilon[1 + (b-a)] + V(\gamma; P) \\ &\le \varepsilon[1 + (b-a)] + V(\gamma) \end{aligned}$$

Letting $\varepsilon \to 0+$, gives

$$\int_a^b |\gamma'(t)|\, dt \le V(\gamma)$$

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which yields equality.

Thm 1.4: Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and suppose that $f : [a,b] \to \mathbb{C}$ is continuous. Then there is a complex number I s.t for every $\varepsilon > 0$, there is $\delta > 0$ s.t when $P = \{t_0 < t_1 < \cdots < t_m\}$ is a partition of $[a,b]$ with $\|P\| = \max\{(t_k - t_{k-1}) : 1 \le k \le m\} < \delta$ then

$$\left| I - \sum_{k=1}^{m} f(\tau_k)\,[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon$$

for whatever choice of pts τ_k, $t_{k-1} \le \tau_k \le t_k$.

Remark: This number I is called the Riemann-Stieljes integral of f with respect to γ over $[a,b]$ and is denoted by

$$I = \int_a^b f\,d\gamma = \int_a^b f(t)\,d\gamma(t).$$

proposition 1.7: Let f and g be continuous functions on $[a,b]$ and let γ and σ be functions of bounded variation on $[a,b]$. Then for any complex scalars α and β:

(a) $\int_a^b (\alpha f + \beta g)\,d\gamma = \alpha \int_a^b f\,d\gamma + \beta \int_a^b g\,d\gamma$

(b) $\int_a^b f\,d(\alpha\gamma + \beta\sigma) = \alpha \int_a^b f\,d\gamma + \beta \int_a^b f\,d\sigma$

proposition 1.8: Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and let $f : [a,b] \to \mathbb{C}$ be continuous. If $a = t_0 < t_1 < \cdots < t_n = b$ then

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$$\int_a^b f\,d\gamma = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f\,d\gamma$$

Thm 1.9: If γ is piecewise smooth and $f: [a,b] \to \mathbb{C}$ is continuous then

$$\int_a^b f\,d\gamma = \int_a^b f(t)\,\gamma'(t)\,dt$$

Defn : If $\gamma: [a,b] \to \mathbb{C}$ is a path if it is continuous. The set $\{\gamma(t): a \le t \le b\}$ is called the trace of γ and is denoted by $\{\gamma\}$. Note $\{\gamma\}$ is always compact. γ is a rectifiable path if γ is a function of bounded variation. To say that γ is rectifiable is to say that γ has finite length and its length is $V(\gamma)$. In particular, if γ is piecewise smooth then γ is rectifiable and its length is $\int_a^b |\gamma'|\,dt$.

Defn 1.12: If $\gamma: [a,b] \to \mathbb{C}$ is a rectifiable path and f is a function defined and continuous on the trace of γ then the (line) integral of f along γ is

$$\int_a^b f(\gamma(t))\,d\gamma(t)$$

This line integral is denoted by $\int_\gamma f = \int_\gamma f(z)\,dz$.

E.g: ① Let $\gamma: [0, 2\pi] \to \mathbb{C}$ be $\gamma(t) = e^{it}$ and $f(z) = \frac{1}{z}$ with $z \neq 0$.

Then $\int_\gamma f\,dz = \int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} e^{-it}(ie^{it})\,dt = 2\pi i$

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② Let $\gamma: [0, 2\pi] \to \mathbb{C}$ be $\gamma(t) = e^{it}$ and

$f(z) = z^m$ where $m \geq 0$ $m \in \mathbb{Z}$. Then

$$\begin{aligned}\int_\gamma f\,dz &= \int_0^{2\pi} e^{imt}\, ie^{it}\,dt \\ &= i\int_0^{2\pi} e^{i(m+1)t}\,dt \\ &= i\int_0^{2\pi} \cos(m+1)t\,dt - \int_0^{2\pi} \sin(m+1)t\,dt \\ &= 0.\end{aligned}$$

③ Fix $a, b \in \mathbb{C}$. Set $\gamma(t) = tb + (1-t)a$ for $0 \leq t \leq 1$.

Then $\gamma'(t) = b - a$. Then when $n \geq 0$,

$$\begin{aligned}\int_\gamma z^n\,dz &= (b-a)\int_0^1 [tb + (1-t)a]^n\,dt \\ &= \frac{1}{n+1}(b^{n+1} - a^{n+1}).\end{aligned}$$

Remark: If $\gamma: [a,b] \to \mathbb{C}$ is rectifiable path and $\varphi: [c,d] \to [a,b]$ is a continuous non-decreasing function with $\varphi(c) = a$, $\varphi(d) = b$. Then $\gamma \circ \varphi$ is also a rectifiable path with $\{\gamma\} = \{\gamma \circ \varphi\}$ and $\int_{\gamma\circ\varphi} f$ is well-defined.

Proposition: If $\gamma: [a,b] \to \mathbb{C}$ is a rectifiable path and $\varphi: [c,d] \to [a,b]$ is a continuous non-decreasing function with $\varphi(c) = a$, $\varphi(d) = b$; then for any function f continuous on $\{\gamma\}$

function f continuous on $\{\sigma\}$

$$\int_\gamma f = \int_{\gamma\circ\varphi} f$$

Defn 1.16: Let $\sigma: [c,d] \to \mathbb{C}$ and $\gamma: [a,b] \to \mathbb{C}$ be rectifiable paths. The path σ is equivalent to γ if there is a function $\varphi: [c,d] \to [a,b]$ which is continuous, strictly increasing, and with $\varphi(c)=a$, $\varphi(d)=b$ s.t $\sigma = \gamma\circ\varphi$. We call the function φ a change of parameter.

A curve is an equivalence class of paths.

Remark: Let $\gamma: [a,b] \to \mathbb{C}$ be a rectifiable path and for $a \le t \le b$, let $|\gamma|(t)$ be $V(\gamma; [a,t])$.

That is,

$$|\gamma|(t) = \sup\left\{\sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| : \{t_0, \cdots, t_n\} \text{ is a partition of } [a,t]\right\}$$

Note $|\gamma|(t)$ is the length of $\gamma(t)$ from a to t.

Clearly $|\gamma|(t)$ is increasing and so $|\gamma|: [a,b] \to \mathbb{R}$ is of bounded variation. If f is continuous on $\{\gamma\}$ define

$$\int_\gamma f\,|dz| = \int_a^b f(\gamma(t))\, d|\gamma|(t)$$

If γ is a rectifiable curve denote by $-\gamma$ the curve defined by $(-\gamma)(t) = \gamma(t)$ for $-b \le t \le -a$. Sometimes we also write γ^{-1} instead of $-\gamma$. Also if $c \in \mathbb{C}$ let $\gamma + c$ denote the curve by $(\gamma+c)(t) = \gamma(t) + c$

proposition 1.17: Let γ be a rectifiable curve and suppose that f is a function continuous on $\{\gamma\}$. Then

(a) $\int_\gamma f = -\int_{-\gamma} f$

(b) $\left|\int_\gamma f\right| \le \int_\gamma |f|\,|dz| \le V(\gamma)\sup[|f(z)| : z\in\{\gamma\}]$.

(c) If $c\in\mathbb{C}$, then $\int_\gamma f(z)\,dz = \int_{\gamma+c} f(z-c)\,dz$.

Thm 1.18: Let G be open in $\mathbb{C}$ and let γ be a rectifiable path in G with initial and end points α and β, respetively. If $f: G\to\mathbb{C}$ is a continuous function with a primitive $F: G\to\mathbb{C}$, then

$$\int_\gamma f = F(\beta)-F(\alpha)$$

Remark: Recall that F is a primitive of f when $F'=f$.

pf of Thm 1.18:

Case I: $\gamma:[a,b]\to\mathbb{C}$ is piecewise smooth.

Then
$$\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,dt\\ &= \int_a^b F'(\gamma(t))\gamma'(t)\,dt\\ &= \int_a^b (F\circ\gamma)'(t)\,dt\\ &= F(\gamma(b))-F(\gamma(a))\\ &= F(\beta)-F(\alpha)\end{aligned}$$

Case II: The general case. See P66 in the book.

A curve $\gamma:[a,b]\to\mathbb{C}$ is said to be closed if $\gamma(a)=\gamma(b)$.

A curve $\gamma : [a,b] \to \mathbb{C}$ is said to be closed if $\gamma(a) = \gamma(b)$.

Corollary 1.22: Let G, γ and f satisfy the same hypothesis as in Thm 1.18. If γ is a closed curve then

$$\int_\gamma f = 0$$

Remark: The fundamental Thm of Calculus says that each continuous function has a primitive. But this is not true in general for functions of a complex variable.

E.g. Let $f(z) = |z|^2 = x^2 + y^2$. Then f has no primitive. Indeed, if F is a primitive of f, then F is analytic. So writing $F = U + iV$, we have

$$F' = \frac{\partial F}{\partial x} = x^2 + y^2$$

$$\Rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = \dfrac{\partial V}{\partial y} = x^2 + y^2 \\ \dfrac{\partial U}{\partial y} = \dfrac{\partial V}{\partial x} = 0 \end{cases}$$

But $\dfrac{\partial U}{\partial y} = 0$ implies that $U(x,y) = u(x)$ for some differentiable function u. But this gives

$$x^2 + y^2 = \frac{\partial U}{\partial x} = u'(x)$$

This is a contradiction.