2. Power series representation of analytic functions

proposition 2.1: Let $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ be a continuous function and define

$$g : [c,d] \to \mathbb{C} \text{ by}$$

$$g(t) = \int_a^b \varphi(s,t)\,ds$$

Then g is continuous. Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a,b]\times[c,d]$ then g is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s,t)\,ds.$$

Remark: This proposition is sometimes called Leibniz's rule.

E.g.: prove $\int_0^{2\pi} \frac{e^{is}}{e^{is}-z}\,ds = 2\pi$ if $|z|<1$

Pf: Let $\varphi(s,t) = \frac{e^{is}}{e^{is}-tz}$ for $0 \le t \le 1$, $0 \le s \le 2\pi$.

Then φ is continuously differentiable.

Hence $g(t) = \int_0^{2\pi} \varphi(s,t)\,ds$ is continuously differentiable.

Note $g(0) = \int_0^{2\pi} 1\,ds = 2\pi$. We will show $g(t)$ is a constant.

$$g'(t) = \int_0^{2\pi} \frac{z e^{is}}{(e^{is}-tz)^2}\,ds$$

ze^{is}

$$g(t) = \int_0 \frac{}{(e^{is}-tz)^2} ds$$

Set $\Phi(s) = zi(e^{is}-tz)^{-1}$. Note $\Phi'(s) = \dfrac{ze^{is}}{(e^{is}-tz)^2}$

Hence
$$g'(t) = \int_0^{2\pi} \Phi'(s)\, ds = \Phi(s)\Big|_0^{2\pi} = 0$$

Thus $g(t)$ is constant.

proposition 2.6: Let $f: G \to \mathbb{C}$ be analytic and suppose $\bar{B}(a;r) \subset G$. If $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$,

then
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\, dw$$

for $|z-a| < r$

Pf: By considering $G_1 = \{\frac{1}{r}(z-a) : z \in G\}$ and the function $g(z) = f(a+rz)$ we see that, without loss of generality, it may be assumed that $a=0$ and $r=1$. That is we may assume that $\bar{B}(0,1) \subset G$.

Fix z, $|z|<1$, it must be shown that
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\, dw$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is}-z}\, ds$$

That is, we want to show that
$$0 = \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is}-z}\, ds - 2\pi f(z)$$
$$= \int_0^{2\pi} \Big[\frac{f(e^{is})e^{is}}{} \quad \Big] ds$$

$$= \int_0^{2\pi} \left[\frac{f(e^{is}) e^{is}}{e^{is} - z} - f(z) \right] ds$$

We will apply Leibniz's rule by letting

$$\varphi(s,t) = \frac{f(z + t(e^{is} - z)) e^{is}}{e^{is} - z} - f(z)$$

for $0 \le t \le 1$ and $0 \le s \le 2\pi$.

Since $|z + t(e^{is} - z)| = |z(1-t) + te^{is}| \le 1$, φ is well-defined and is continuously differentiable.

Let $g(t) = \int_0^{2\pi} \varphi(s,t)\, ds$. Then g is continuously differentiable.

Claim 1: $g(0) = 0$.

Pf:
$$g(0) = \int_0^{2\pi} \varphi(s,0)\, ds$$
$$= \int_0^{2\pi} \left[\frac{f(z) e^{is}}{e^{is} - z} - f(z) \right] ds$$
$$= f(z) \int_0^{2\pi} \frac{e^{is}}{e^{is} - z}\, ds - 2\pi f(z)$$
$$= 0.$$

Claim 2: $g' = 0$ for $0 < t \le 1$. Thus g is constant.

Pf: By Leibniz's rule, $g'(t) = \int_0^{2\pi} \varphi_2(s,t)\, ds$, where

$$\varphi_2(s,t) = e^{is} f'(z + t(e^{is} - z))$$

However, for $0 < t \le 1$, we have that

$\Phi(s) = -it^{-1} f(z + t(e^{is} - z))$ is a primitive of $\varphi_2(s,t)$.

So $g'(t) = \Phi(2\pi) - \Phi(0) = 0$ for $0 < t \le 1$. Thus g

So $g'(t) = \Phi(2\pi) - \Phi(0) = 0$ for $0 < t \le 1$. Thus g must be a constant.

Remark: We will use proposition 2.6 to prove every analytic function has the power series expansion.

Lemma 2.7: Let γ be a rectifiable curve in $\mathbb{C}$ and suppose that F_n and F are -inuous functions on $\{\gamma\}$. If $F = u\text{-}\lim F_n$ on $\{\gamma\}$ then

$$\int_\gamma F = \lim \int_\gamma F_n.$$

Theorem 2.8: Let f be analytic in $B(a;R)$; then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < R$ where

$$a_n = \frac{1}{n!} f^{(n)}(a) \text{ and}$$

this series has radius of convergence $\ge R$.

Pf: Let $0 < r < R$ so that $\overline{B}(a;r) \subset B(a;R)$. If $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$, then by proposition 2.6,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw \text{ for } |z-a| < r$$

But, since $|z-a| < r$ and w is on the circle $\{\gamma\}$.

$$\frac{|f(w)||z-a|^n}{|w-a|^{n+1}} \le \frac{M}{r}\left(\frac{|z-a|}{r}\right)^n$$

where $M = \max\{|f(w)| : |\ -a| = r\}$. Since $\frac{|z-a|}{r} < 1$, the Weierstrass M-test gives that the Weierstrass M-test gives that

$$\sum f(w) \frac{(z-a)^n}{(w-a)^{n+1}} \text{ converges uniformly}$$

for w on $\{\gamma\}$.

Note when $|z-a|<r$ and $|w-a|=r$

Then
$$\frac{1}{w-z} = \frac{1}{w-a} \cdot \frac{1}{1-\left[\frac{z-a}{w-a}\right]} = \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n . \quad (*)$$

as $|z-a|<r=|w-a|$

Multiple both sides of $(*)$ by $f(w)/2\pi i$ and integrate, we get,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n dw$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw\right] (z-a)^n \qquad (**)$$

we set
$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw$$

then a_n is independent of z. So $(**)$ is a power series which converges for $|z-a|<r$.