4.3 Zeros of an analytic function

Recall

① Euclidean Divison theorem

If $s(z)$ and $p(z)$ are two polynomials, then $\exists$ two polynomials $q(z)$ and $r(z)$ such that

$$s(z) = p(z)q(z) + r(z)$$

with $\deg r(z) < \deg p(z)$.

② If $s(a)=0$, then $s(z) = (z-a)q(z)$ for some polynomial q.

Moreover, $\exists$ $m \geq 1$ s.t

$$s(z) = (z-a)^m g(z) \quad \text{with } g(a) \neq 0.$$

$$\deg g = \deg s - m.$$

Defn 3.1: If $f: G \to \mathbb{C}$ is analytic and a in G satisfies $f(a)=0$. then a is a zero of f. We say it is of multiplicity $m \geq 1$. If there is an analytic function $g: G \to \mathbb{C}$ s.t

$$f(z) = (z-a)^m g(z) \quad \text{where } g(a) \neq 0.$$

Defn 3.2: An entire function is a function which is defined and analytic in $\mathbb{C}$

proposition 3.3: If f is an entire function then f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with infinite radius of convergence.

Remark: entire functions can be considered as polynomials of "infinite degree". Note the only bounded

Remark: entire functions can be considered as polynomials
of "infinite degree". Note the only bounded polynomials are constant polynomials. Indeed, If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with $n \geq 1$,
then $p(z) = \lim_{z\to\infty} z^n[1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}] = \infty$

Theorem 3.4 (Liouville's Theorem) If f is bounded entire function then f is constant.

Pf: Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We will show $f'(z) = 0$ for all $z \in \mathbb{C}$. To do this use Cauchy's Esimate. Since f is analytic in any disk $B(z; R)$ we have that $|f'(z)| \leq \frac{M}{R}$. Let $R \to +\infty$, we have, $f'(z) = 0$

Thm 3.5. (Fundamental Theorem of Algebra)

If $p(z)$ is non constant polynomial then there $\exists$ $a \in \mathbb{C}$ s.t $p(a) = 0$.

Pf: Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Let $f(z) = \frac{1}{p(z)}$. It's well-defined for all $z \in \mathbb{C}$ $\Rightarrow$ f is an entire function. If p is not constant, then $\lim_{z\to\infty} p(z) = \infty$. and $\lim_{z\to\infty} f(z) = 0$. Thus $\exists$ $R >> 1$, s.t $|f(z)| < 1$ if $|z| > R$. But f is continuous on $\overline{B}(0; R)$, so there is $M > 0$ s.t $|f(z)| \leq M$ for $|z| \leq R$. Hence f is bounded and, therefore, it is constant by Liouville's theorem. It follows that p must be constant, contradicting our assumption.

Corollary 3.6: If $p(z)$ is a polynomial and $a_1, \cdots, a_m$ are its zeros with a_j having multiplicity k_j then
$$p(z) = c(z-a_1)^{k_1} \cdots (z-a_m)^{k_m} \text{ for some constant } c$$
and $k_1 + \cdots + k_m$ is the degree of p.

Theorem 3.7: Let G be a connected open set and let $f: G \to \mathbb{C}$ be an analytic function. Then the following are equivalent:

(a) $f \equiv 0$

(b) there is a pt $a \in G$ s.t $f^{(n)}(a) = 0$ for each $n \geq 0$

(c) $\{z \in G : f(z) = 0\}$ has a limit pt in G

pf: (1) Clearly (a) $\Rightarrow$ (b), (c)

(2) (c) $\Rightarrow$ (b).

Let $a \in G$ and a limit pt of $Z = \{z \in G : f(z) = 0\}$.
Let $R > 0$ be s.t $B(a; R) \subset G$. Since a is a limit pt of Z and f is continuous it follows that $f(a) = 0$.

We need to prove $f^{(n)}(a) = 0$ for all $n \geq 0$. Suppose Not. Then there is an integer $n \geq 1$ s.t
$$f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0 \text{ and}$$
$$f^{(n)}(a) \neq 0.$$

Expanding f is power series about a gives that
$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k$$
for $|z-a| < R$. If $g(z) = \sum_{k=n}^{\infty} a_k (z-a)^{k-n}$ then g is analytic in $B(a; R)$, $f(z) = (z-a)^n g(z)$, and

then g is analytic in $B(a;R)$, $f(z) = (z-a)^n g(z)$, and $g(a) = a_n \neq 0$. Since g is analytic (and therefore continuous) in $B(a;R)$ we can find an r, $0<r<R$, s.t $g(z) \neq 0$ for $|z-a|<r$. This gives $0=(b-a)^n g(b)$ and so $g(b)=0$, a contradiction. Hence no such integer n can exist, this proved (b).

(2) (b) $\Rightarrow$ (a): Let $A = \{z \in G : f^{(n)}(z) = 0 \text{ for all } n \geq 0\}$. From the hypothesis of (b) we have that $A \neq \phi$. We will show A is both open and closed in G. Then by the connectedness of G it follows that $A = G$.

Note the closedness of A can be seen from the continuity of $f^{(n)}(z)$.

To see A is open, Let $a \in A$. First $\exists R>0$ s.t $B(a;R) \subset G$. Then $f(z) = \sum a_n (z-a)^n$ for $|z-a|<R$ where $a_n = \frac{1}{n!} f^{(n)}(a) = 0$ for every $n \geq 0$. Hence $f(z)=0$ in $B(a;R)$. Consequently, $B(a;R) \subset A$. Thus A is open.

Corollary 3.8: If f and g are analytic on a region G then $f \equiv g$ iff $\{z \in G : f(z) = g(z)\}$ has a limit pt in G

Corollary 3.9: If f is analytic on an open connected set G and f is not identically zero then for each a in G with $f(a)=0$ there is an integer $n \geq 1$ and an analytic function $g: G \to \mathbb{C}$ such that $g(a) \neq 0$ and $f(z) = (z-a)^n g(z)$ for all z in G. That is, each zero of f has finite multiplicity.

Corollary 3.10: If $f: G \to \mathbb{C}$ is analytic and not constant, $a \in G$, and

Corollary 3.10: If $f: G \to \mathbb{C}$ is analytic and not constant, $a \in G$, and $f(a) = 0$, then $\exists$ an $R > 0$ s.t $B(a;R) \subset G$ and $f(z) \neq 0$ for $0 < |z-a| < R$.

Remark. The zeros of f are isolated.

Thm 3.11: (Maximum Modulus Theorem) If G is a region and $f: G \to \mathbb{C}$ is an analytic function s.t there is a pt $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

Pf: Let $\overline{B(a;r)} \subset G$, $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$; By Cauchy's Integral formula,

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a}\, dw$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt$$

Hence

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{it})|\, dt \leq |f(a)|$$

Since $|f(a+re^{it})| \leq |f(a)|$ for all t. This gives that.

$$0 = \int_0^{2\pi} [\,|f(a)| - |f(a+re^{it})|\,]\, dt$$

but since the integral is non-negative it follows that $|f(a)| = |f(a+re^{it})|$ for all t. Moreover, since r was arbitrary, we have f maps any disk $B(a;R) \subset G$ into the circle $\{z \in \mathbb{C}: |z| = |f(a)|\}$. But this implies that f is constant on $B(a;R)$. Hence $f(z) = f(a)$ for $z \in B(a;R)$. By Corollary 3.8, $f \equiv f(a)$.

4. The index of a closed curve

Recall: $\int (z-a)^{-1} dz = 2\pi i n$ if $\gamma(t) = a + e^{2\pi i n t}$, $0 \leq t \leq 1$.

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The next result shows that this is not particular to the path γ.

Proposition 4.1: If $\gamma: [0,1] \to \mathbb{C}$ is a closed rectifible curve and $a \notin \{\gamma\}$ then

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

is an integer.

Pf: We will only prove for the case when γ is smooth.

In this case define $g: [0,1] \to \mathbb{C}$ by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a}\, ds$$

Hence, $g(0) = 0$ and $g(1) = \int_\gamma \frac{1}{z-a}dz$. We also have

$$g'(t) = \frac{\gamma'(s)}{\gamma(s)-a} \quad \text{for } 0 \le t \le 1.$$

But this gives,

$$\frac{d}{dt} e^{-g}(\gamma - a)$$
$$= e^{-g}\gamma' - g'e^{-g}(\gamma-a)$$
$$= e^{-g}[\gamma' - g'(\gamma-a)]$$
$$= 0$$

So $e^{-g}(\gamma-a)$ is constant $\Longrightarrow$

$$e^{-g(0)}(\gamma(0)-a) = \gamma(0) - a = e^{-g(1)}(\gamma(1)-a)$$

Since $\gamma(0) = \gamma(1)$ we have that $e^{-g(1)} = 1$, i.e.,

$$g(1) = 2\pi i k \text{ for some integer } k.$$

Defn 4.2: If γ is a closed rectifible curve in $\mathbb{C}$ then for $a \notin \{\gamma\}$

$$n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma (z-a)^{-1}dz$$

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is called the index of γ with respect to the pt a. It is also sometimes called the winding number of γ around a.

Recall: Let $\gamma, \sigma : [0,1] \to \mathbb{C}$ be curves. define

$$(-\gamma)(t) \triangleq \gamma(1-t)$$

$$(\gamma+\sigma)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \sigma(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

proposition 4.3: If γ and σ are closed rectifible curves having the same initial pts then

(a) $n(\gamma; a) = -n(-\gamma; a)$ for every $a \notin \{\gamma\}$

(b) $n(\gamma+\sigma; a) = n(\gamma; a) + n(\sigma; a)$ for every $a \notin \{\gamma\} \cup \{\sigma\}$.

E.g: Let $\gamma(t) = a + e^{2\pi i n t}$ for $0 \le t \le 1$. Then $n(\gamma; a) = n$. Indeed,

If $|b-a| < 1$, then $n(\gamma; b) = n$

If $|b-a| > 1$, then $n(\gamma; b) = 0$.

Remark: $n(\gamma; b)$ measures the number of times γ wraps around b.

Let γ be a closed rectifible curve and consider the open set $G = \mathbb{C} - \{\gamma\}$. since $\{\gamma\}$ is compact $\{z : |z| > R\} \subset G$ for some large R. This says that G has one, and only one, unbounded component.

Thm 4.4: Let γ be a closed rectifible curve in $\mathbb{C}$. Then $n(\gamma; a)$ is constant for a belonging to a component of $G = \mathbb{C} - \{\gamma\}$. Also, $n(\gamma; a) = 0$ for a belonging to the unbounded component of G.

Also, $n(\gamma; a) = 0$ for a belonging to the unbounded component of G.

Proof: Define $f: G \to \mathbb{C}$ by $f(a) = n(\gamma; a)$. It will be shown that f is continuous. If this is done, it follows that $f(D)$ is connected for each component D of G. But since $f(G)$ is contained in the set of integer it follows that $f(D)$ reduced to a single pt. That is, f is constant on D.

Next we show f is continuous, recall that the components of G are open. Fix $a \in G$ and let $r = d(a, \{\gamma\})$. If $|a-b| < \delta < \frac{1}{2}r$ then

$$|f(a) - f(b)| = \frac{1}{2\pi}\left|\int_\gamma [(z-a)^{-1} - (z-b)^{-1}]\,dz\right|$$
$$= \frac{1}{2\pi}\left|\int_\gamma \frac{(a-b)}{(z-a)(z-b)}\,dz\right|$$
$$\leq \frac{|a-b|}{2\pi}\int_\gamma \frac{|dz|}{|z-a||z-b|}$$

But for $|a-b| < \frac{1}{2}r$ and z on $\{\gamma\}$ we have that $|z-a| \geq r > \frac{1}{2}r$ and $|z-b| > \frac{1}{2}r$. It follows that

$$|f(a) - f(b)| < \frac{2\delta}{\pi r^2} V(\gamma).$$

So if $\varepsilon > 0$ is given then, by choosing δ to be smaller than $\frac{1}{2}r$ and $(\pi r^2 \varepsilon)/2V(\gamma)$, we see that f must be continuous.

Now let U be the unbounded component of G. Recall there is $R > 0$ s.t. $U \supset \{z : |z| > R\}$. If $\varepsilon > 0$, choose a with $|a| > R$; $|z-a| > (2\pi\varepsilon)^{-1} V(\gamma)$ uniformly for z on $\{\gamma\}$, then $|n(\gamma;a)| < \varepsilon$. That is, $n(\gamma; a) \to 0$

with $|a| > R$; $|z-a| > (2\pi\varepsilon)^{-1} V(\gamma)$ uniformly for z on $\{\gamma\}$; then $|n(\gamma;a)| < \varepsilon$. That is, $n(\gamma;a) \to 0$ as $a \to \infty$. Since $n(\gamma;a)$ is constant, it must be zero.