4.5 Cauchy's Theorem and Integral Formula

Recall: If $G = B(a;R)$, then $\int_\gamma f = 0$ for any analytic function f on G and any closed rectifible curve γ in G.

Remark: This is not true for all regions G. For example, if $G = \mathbb{C} - \{0\}$, $f(z) = z^{-1}$, $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$ then

$$\int_\gamma f = 2\pi i$$

Q: Fix a region G and an analytic function f on G. Find a condition on a closed rectifiable curve γ such that $\int_\gamma f = 0$?

Lemma 5.1. Let γ be a rectifible curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \ge 1$ let $F_m(z) = \int_\gamma \varphi(w)(w-z)^{-m}\,dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} - \{\gamma\}$ and $F_m'(z) = mF_{m+1}(z)$.

Pf: Claim: F_m is continuous in $\mathbb{C} - \{\gamma\}$.

Pf of Claim: Exercise.

Next note

① Since $\{\gamma\}$ is compact $\Rightarrow \varphi$ is bounded on $\{\gamma\}$.

② Since

$$A^m - B^m = (A-B)\sum_{k=1}^{m} A^{m-k}B^{k-1}$$

$$A^m - B^m = (A-B)\sum_{k=1}^{\cdots} A^{m-k}B^{k-1}$$

$$\Rightarrow \frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} = \left[\frac{1}{w-z} - \frac{1}{w-a}\right]\sum_{k=1}^{m}\frac{1}{(w-z)^{m-k}}\frac{1}{(w-a)^{k-1}}$$

$$= (z-a)\left[\frac{1}{(w-z)^m(w-a)} + \frac{1}{(w-z)^{m-1}(w-a)^2} + \cdots + \frac{1}{(w-z)(w-a)^m}\right]$$

Now fix a in $G = \mathbb{C} - \{\gamma\}$ and let $z \in G$, $z \neq a$. we have

$$\frac{F_m(z) - F_m(a)}{z-a} = \int_\gamma \frac{\varphi(w)(w-a)^{-1}}{(w-z)^m}dw + \cdots + \int_\gamma \frac{\varphi(w)(w-a)^{-m}}{w-z}dw$$

Note each integral defines a continuous function of z, z in G.

Hence $z \to a$, we have

$$F_m'(a) = \int_\gamma \frac{\varphi(w)}{(w-a)^{m+1}}dw + \cdots + \int_\gamma \frac{\varphi(w)}{(w-a)^{m+1}}dw$$

$$= mF_{m+1}(a)$$

Thm 5.4 (Cauchy's Integral Formula; First Version)

Let G be an open subset of the plane and $f: G \to \mathbb{C}$ an analytic function. If γ is a closed rectifible curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$, then for a in $G - \{\gamma\}$

$$n(\gamma; a) f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$$

Pf: Define $\varphi: G \times G \to \mathbb{C}$ by $\varphi(z,w) = \dfrac{f(z) - f(w)}{z-w}$ if $z \neq w$ and

$\varphi(z,z) = f'(z)$

Claim: φ is continuous. For each $w \in G$, $z \to \varphi(z,w)$ is analytic.

Pf of claim: Exercise.

Let $H = \{w \in \mathbb{C} : n(\gamma; w) = 0\}$. Since $n(\gamma; w)$ is a continuous integer-valued function of w, H is open. Note $\mathbb{C} - G \subset H$. we have $H \cup G = \mathbb{C}$.

Define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \int_\gamma \varphi(z,w)\,dw$ if $z \in G$ and $g(z) = \int_\gamma (w-z)^{-1} f(w)\,dw$ if $z \in H$. If $z \in G \cap H$ then

$$\int_\gamma \varphi(z,w)\,dw = \int_\gamma \frac{f(w)-f(z)}{w-z}\,dw$$
$$= \int_\gamma \frac{f(w)}{w-z}\,dw - f(z)\, n(\gamma; z)\, 2\pi i$$
$$= \int_\gamma \frac{f(w)}{w-z}\,dw$$

Hence g is well-defined function.

By Lemma 5.1 g is analytic on H and by Exercise 2.2, g is analytic on G. Thus g is an entire function. But Thm 4.4 implies that H contains the unbounded component of $\mathbb{C} - \{\gamma\}$. Thus

$$g(z) = \int_\gamma \frac{f(w)}{w-z}\,dw \quad \text{if } z \text{ is large}$$

As f is bdd on $\{\gamma\}$ and $\lim\limits_{z\to\infty} \frac{1}{w-z} = 0$ uniformly for w in $\{\gamma\}$ $\Longrightarrow$

$$\lim g(z) = \lim \int_\gamma \frac{f(w)}{w-z}\,dw = 0.$$

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \int_\gamma \frac{f(w)}{w-z}\,dw = 0.$$

We conclude that g is bdd entire function.

By Liouville's thm, g is constant. But $\lim_{z\to\infty} g(z) = 0$.

we have $g \equiv 0$. Let $a \in G - \{\gamma\}$. Then

$$0 = \int_\gamma \frac{f(w) - f(a)}{w - a}\,dw$$

$$= \int_\gamma \frac{f(w)}{w-a}\,dw - f(a)\int_\gamma \frac{dw}{w-a}$$

This proves the thm.

Thm 5.6 (Cauchy's Integral Formula, Second Version)

Let G be an open subset of the plane and $f: G \to \mathbb{C}$ analytic function. If $\gamma_1, \cdots, \gamma_m$ are closed rectifible curves in G such that $n(\gamma_1; w) + \cdots + n(\gamma_m, w) = 0$ for all w in $\mathbb{C} - G$, then for a in $G - \bigcup_{k=1}^{m} \{\gamma_k\}$

$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a}\,dz$$

Pf: Similar to the pf of Thm 5.4.

Thm 5.7 (Cauchy's Thm, first Version)

Let G be an open subset of the plane and $f: G \to \mathbb{C}$

Let G be an open subset of the plane and $f: G \to \mathbb{C}$ an analytic function. If $\gamma_1, \cdots, \gamma_m$ are closed rectifible curves in G such that $n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$ for all w in $\mathbb{C} - G$ then

$$\sum_{k=1}^{m} \int_{\gamma_k} f = 0$$

Pf: Substitute $f(z)(z-a)$ for f in Thm 5.6.

E.g: Let $G = \{z : R_1 < |z| < R_2\}$ and define curve γ_1 and γ_2 in G by $\gamma_1(t) = r_1 e^{it}$, $\gamma_2(t) = r_2 e^{-it}$ for $0 \le t \le 2\pi$, where $R_1 < r_1 < r_2 < R_2$. If $|w| \le R_1$, $n(\gamma_1; w) = 1 = -n(\gamma_2; w)$; if $|w| \ge R_2$, then $n(\gamma_1; w) = n(\gamma_2; w) = 0$

Hence $n(\gamma_1; w) + n(\gamma_2; w) = 0$ for all w in $\mathbb{C} - G$.

Thm 5.8: Let G be an open subset of the plane and $f: G \to \mathbb{C}$ an analytic function. If $\gamma_1, \cdots, \gamma_m$ are closed rectifible curves in G s.t $n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$ for all w in $\mathbb{C} - G$. Then for $a \in G - \{\gamma\}$ and $k \ge 1$

$$f^{(k)}(a) \sum_{j=1}^{m} n(\gamma_j; w) = k! \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}}\, dz$$

Pf: This follows immediately by differentiating both sides of the formula in Thm 5.6 and applying Lemma 5.1.

Corollary 5.9: Let G be an open set and $f: G \to \mathbb{C}$ an analytic function. If γ is a closed rectifible curve in G s.t $n(\gamma; w) = 0$ for all $w \in \mathbb{C} - G$ then for a in $G - \{\gamma\}$

$$f^{(k)}(a)\, n(\gamma; a) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{k+1}}\, dz$$

Thm 5.10 (Morera's Thm)

Let G be a region and let $f: G \to \mathbb{C}$ be a continuous function s.t $\int_T f = 0$ for every triangular path T in G, then f is analytic in G.

Pf: Note it suffices to show f is analytic on each open disk contained in G. Hence, WLOG, we may assume G to be an open disk. That is, assume $G = B(a; R)$.

We will prove that f has a primitive. For $z \in G$ define $F(z) = \int_{[a,z]} f$. Fix $z_0 \in G$, then for any pt $z \in G$ the hypothesis gives that $F(z) = \int_{[a,z_0]} f + \int_{[z_0,z]} f$

Hence $$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0, z]} f$$

This gives

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0, z]} f - f(z_0)$$

$$= \frac{1}{z - z_0} \int_{[z_0, z]} [f(w) - f(z_0)]\, dw$$

$$z - z_0 \int_{[z_0, z]} [f(w) - f(z_0)]\, dw$$

But by taking absolute values

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \sup \left\{ |f(w) - f(z_0)| : w \in [z, z_0] \right\}$$

which shows that

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0).$$