6. The homotopic version of Cauchy's Theorem and Simply Connectivity

Defn 6.1: Let $\gamma_0, \gamma_1 : [0,1] \to G$ be closed rectifible curves in region G, then γ_0 is homotopic to γ_1 in G if there is a continuous function $\Gamma : [0,1]\times[0,1] \to G$ s.t

$$\begin{cases} \Gamma(s,0) = \gamma_0(s) \text{ and } \Gamma(s,1) = \gamma_1(s) & (0 \le s \le 1) \\ \Gamma(0,t) = \Gamma(1,t) & (0 \le t \le 1) \end{cases} \quad (*)$$

If γ_0 is homotopic to γ_1 in G write $\gamma_0 \sim \gamma_1$.

Remark: "$\sim$" is an equivalence relation

E.g.: Let $G = B(a;R)$ and let $\gamma : [0,1] \to G$ be a closed rectifible curve. Fix $b \in G$. Set

$$\Gamma(s,t) = tb + (1-t)\gamma(s), \quad 0 \le s \le 1, \quad 0 \le t \le 1.$$

Then $\Gamma(s,t) \in G$

Note $\Gamma(s,0) = \gamma(s)$; $\Gamma(s,1) = b$.

Thus γ is homotopic to a constant curve

This process does NOT work if $G = B(a;R) \setminus \{a\}$.

Defn: A set G is convex if given any two points $a, b \in G$, the line segment joining a and b, $[a,b]$, lies entirely in G. The set G is star shaped if there is a point $a \in G$ s.t for each $z \in G$, the line segment $[a,z]$ lies entirely in G. In this case, we say G is a-star shaped

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Remark: ① If G is a-star shaped and z and w are pts in G then $[z, a, w]$ is polygon in G connecting z and w. Hence, each star shaped set is connected.

② convex $\Rightarrow$ star-shaped but the converse is not true.

Proposition 6.4: Let G be an open set which is a-star shaped. Every closed rectifible curve in G is homotopic to a constant curve.

Defn 6.5: If γ is a closed rectifible curve in G then γ is homotopic to zero ($\gamma \sim 0$) if γ is homotopic to a constant curve.

Thm 6.6 (Cauchy's Thm; Second Version)

If $f: G \to \mathbb{C}$ is an analytic function and γ is a closed rectifible curve in G such that $\gamma \sim 0$, then

$$\int_\gamma f = 0$$

Pf: We just explain the idea. It suffices to prove that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} - G$ whenever $\gamma \sim 0$.

Let $\gamma_1 = \gamma$ and let γ_0 be the constant curve s.t $\gamma_1 \sim \gamma_0$. Let Γ satisfy (*) and define

$h(t) = n(\gamma_t; w)$, where $\gamma_t = \Gamma(s, t)$ and

$$h(t) = n(\gamma_t; w), \text{ where } \gamma_t = \Gamma(s, t) \text{ and}$$
w is fixed in $\mathbb{C} - G$.

Step 1: show h is continuous on $[0,1]$.

Then since h is integer valued and $h(0) = 0$ it must be that $h(t) \equiv 0$. In particular,

$$n(\gamma; w) = 0 \text{ for all } w \in \mathbb{C} - G.$$

Thm 6.7 (Cauchy's Thm, third Version)

If γ_0 and γ_1 are closed rectifible curves in G and $\gamma_0 \sim \gamma_1$ then
$$\int_{\gamma_0} f = \int_{\gamma_1} f$$
for every function f analytic on G.

Pf: Read P90-92 in the book.

Corollary 6.10: If γ is a closed rectifible curve in G s.t $\gamma \sim 0$ then $n(\gamma; w) = 0$ for all $w \in \mathbb{C} - G$.

Defn: An open set is simply connected if G is connected and every closed curve in G is homotopic to zero.

Thm 6.15: If G is simply connected then $\int_\gamma f = 0$ for every closed rectifible curve and every analytic function f in G.

Defn 6.11: If $\gamma_0, \gamma_1 : [0,1] \to G$ are two rectifible curves

Defn 6.11: If $\gamma_0, \gamma_1 : [0,1] \to G$ are two rectifible curves in G s.t $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$ then we say γ_0 and γ_1 are fixed-end-point (FEP homotopic) if there is a continuous map $\Gamma : I^2 \to G$ s.t

$$\Gamma(s,0) = \gamma_0(s) \quad \Gamma(s,1) = \gamma_1(s)$$
$$\Gamma(0,t) = a \quad \Gamma(1,t) = b$$

for $0 \le s, t \le 1$

Thm 6.13: (Independence of path Thm)

If γ_0 and γ_1 are two rectifible curves in G from a to b and γ_0 and γ_1 are FEP homotopic then

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

for any analytic function f in G.

Pf: (Idea) let $\gamma = \gamma_0 - \gamma_1$. Then prove $\gamma \sim 0$ in G

See P93 in the book for details.

Corollary 6.16: If G is simply connected and $f: G \to \mathbb{C}$ is analytic in G then f has a primitive in G.

Pf: Fix a pt $a \in G$ and let γ_1, γ_2 be any two rectifible curves in G from a to a pt $z \in G$. Then we have

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

Hence we can get a well-defined function $F: G \to \mathbb{C}$

Hence we can get a well-defined function $F: G \to \mathbb{C}$ by setting $F(z) = \int_\gamma f$ where γ is any rectifible curve from a to z.

Claim: F is a primitive of f.

Pf of claim: Fix $z_0 \in G$. Let $r>0$ s.t $B(z_0, r) \subset G$, then let γ be a path from a to z_0. For z in $B(z_0; r)$ let $\gamma_z = \gamma + [z_0, z]$. Then

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0, z]} f$$

Now proceed as in the pf of Morera's Thm to show that $F'(z_0) = f(z_0)$.

Corollary 6.17: Let G be simply connected and let $f: G \to \mathbb{C}$ be an analytic function s.t $f(z) \neq 0$ for any $z \in G$. Then there is an analytic function $g: G \to \mathbb{C}$ s.t

$$f(z) = e^{g(z)}.$$

Pf: Since $f \neq 0$ everywhere, $\frac{f'}{f}$ is analytic on G. Hence by Corollary 6.16, it has a primitive g_1. Let $h = e^{g_1}$. Then h is analytic and never vanishes. So $\frac{f}{h}$ is analytic and

$$\left(\frac{f}{h}\right)' = \frac{hf' - h'f}{h^2}$$

Note $h' = g_1' h \Rightarrow \frac{h'}{h} = g_1' = \frac{f'}{f} \Rightarrow \left(\frac{f}{h}\right)' = 0$

Note $\quad h' = g_1 h \quad \Rightarrow \quad \frac{h'}{h} = g_1 = \frac{f'}{f} \Rightarrow (h/$ …

Thus $\frac{f}{h} \equiv c$ is a constant. $\Longrightarrow$

$$f = ch = ce^{g_1} = e^{g_1 + c'}$$

(Here as $f \neq 0 \Rightarrow c \neq 0$)

Then we just need to set $g = g_1 + c'$.