7. Counting zeros; Open mapping Theorem.

Recall: If a nonconstant function f has a zero at $z=a$, we have

$$f(z) = (z-a)^m g(z)$$

where g is analytic and $g(a) \neq 0$.

Suppose G is a region and $f: G \to \mathbb{C}$ analytic. Assume f has zeros $a_1, \cdots, a_m$ in G. Then

$$f(z) = (z-a_1) \cdots (z-a_m) g(z)$$ where g analytic on G,

with $g(z) \neq 0$ on G. This implies

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}$$

for $z \neq a_1, \cdots, a_m$.

Thm 7.2 Let G be a region and let f be an analytic function on G with zeros $a_1, \cdots, a_m$ (repeated according to multiplicity). If γ is a closed rectifible curve in G which does not pass through any point a_k and if $\gamma \approx 0$ in G then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{n} n(\gamma; a_k)$$

Pf: If $g(z) \neq 0$ for any $z \in G$ then $\frac{g'}{g}$ is analytic in G. Since $\gamma \approx 0$ in G, Cauchy's Theorem implies

Since $\gamma \approx 0$ in G, Cauchy's Theorem implies

$$\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0.$$

We have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a_1}\,dz + \cdots + \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a_m}\,dz + \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz$$

$$= n(\gamma;a_1) + \cdots + n(\gamma;a_m)$$

$$= \sum_{k=1}^{m} n(\gamma;a_k)$$

Corollary 7.3: Let f, G and γ be as in Thm 7.2 except that $a_1, \cdots, a_m$ are pts in G that satisfy $f(z)=\alpha$; then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}\,dz = \sum_{k=1}^{m} n(\gamma;a_k)$$

E.g. Compute $\int_\gamma \frac{(2z+1)}{z^2+z+1}\,dz$ where $\gamma = 2e^{it}$, $0 \le t \le 2\pi$

Note $z^3-1 = (z-1)(z^2+z+1) = (z-1)(z-w_1)(z-w_2)$

where $w_1 = e^{\frac{2\pi i}{3}}$, $w_2 = e^{\frac{4\pi i}{3}}$.

$z^2+z+1 = (z-w_1)(z-w_2)$

Then let $f = z^2+z+1 \Rightarrow$

$$\int_\gamma \frac{2z+1}{z^2+z+1} = \int_\gamma \frac{f'}{f} = 2\pi i \sum_{j=1}^{2} n(\gamma; w_j) = 4\pi i$$

Remark: Let $\gamma : [0,1] \to G$ be a closed rectifible curve in $\mathbb{C}$, Remark: Let $\gamma : [0,1] \to G$ be a closed rectifible curve in $\mathbb{C}$, $\gamma \approx 0$. Assume $f: G \to \mathbb{C}$ analytic. Then $f \circ \gamma = \sigma$ is a closed rectifible curve in $\mathbb{C}$. Suppose that α is complex number with $\alpha \notin \{\sigma\} = f(\{\gamma\})$, and let us calculate $n(\sigma; a)$. We get

$$n(\sigma;\alpha) = \frac{1}{2\pi i}\int_\sigma \frac{dw}{w-\alpha}$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}\,dz$$

$$= \sum_{k=1}^{m} n(\gamma; a_k)$$

where a_k are the pts in G with $f(a_k) = \alpha$.

Thm 7.4: Suppose f is analytic in $B(a;R)$ and let $\alpha = f(a)$. If $f(z) - \alpha$ has a zero of order m at $z = a$, then there is an $\varepsilon > 0$ and $\delta > 0$ such that for $0 < |\xi - \alpha| < \delta$, the eqn $f(z) = \xi$ has exactly m simple roots in $B(a;\varepsilon)$.

Remark: A simple root of $f(z) = \xi$ is a zero of $f(z) - \xi$ of multiplicity 1. Note this thm implies $B(\alpha;\delta) \subset f(B(a;\varepsilon))$. Also, the hypothesis $f(z) - \alpha$ has a zero of finite multiplicity guarantees that f is NOT constant.

Pf: Since the zeros of f are isolated we can choose $\varepsilon > 0$

Pf: Since the zeros of f are isolated we can choose $\varepsilon > 0$ s.t $\varepsilon < \frac{1}{2}R$, $f(z) = \alpha$ has no solutions in $\{z : 0 < |z-a| < 2\varepsilon\}$, and $f'(z) \neq 0$ if $0 < |z-a| < 2\varepsilon$

Let $\gamma(t) = a + \varepsilon e^{2\pi i t}$, $0 \le t \le 1$. Put $\sigma = f \circ \gamma$.

Now $\alpha \notin \{\sigma\}$; So there is $\delta > 0$ s.t $B(\alpha;\delta) \cap \{\sigma\} = \phi$.

Thus, $B(\alpha;\delta)$ is contained in some component of $\mathbb{C} - \{\sigma\}$; that is, $|\alpha - \xi| < \delta$ implies

$$n(\sigma;\alpha) = n(\sigma;\xi) = \sum_{k=1}^{p} n(\gamma; z_k(\xi))$$

But since $n(\gamma;z)$ must be either 0 or 1, we have that there are exactly m solutions to the eqn $f(z) = \xi$ inside $B(a;\varepsilon)$. Since $f'(z) \neq 0$ for $0 < |z-a| < \varepsilon$, each of these roots for $\xi \neq a$ must be simple.

Thm 7.5 (open mapping Thm) Let G be a region and suppose that f is a nonconstant analytic function on G. Then for any open set $U \subset G$, $f(U)$ is open.

Pf: Fix $a \in U$ and write $\alpha = f(a)$. It suffices to show that there $\exists\ \delta > 0$ s.t $B(\alpha;\delta) \subset f(U)$. But the above thm implies there $\exists\ \varepsilon > 0$ and $\delta > 0$ s.t $B(a;\varepsilon) \subset U$ and $f(B(a;\varepsilon)) \supset B(\alpha;\delta)$

Remark: (1). A map $f: X \to \Omega$ between two metric spaces is called an open map if it maps open sets to open

Remark: (1). A map $f: X \to \Omega$ between two metric spaces is
Called an open map if it maps open sets to open sets

(2) Note if f is one-to-one and onto, then $f^{-1}: \Omega \to X$ is well-defined.

Exercise: f^{-1} continuous $\Leftrightarrow$ f is open

Corollary 7.6: Suppose $f: G \to \mathbb{C}$ is one-to-one, analytic, and $f(G) = \Omega$. Then $f^{-1}: \Omega \to \mathbb{C}$ is analytic and

$$(f^{-1})'(w) = \frac{1}{f'(z)} \quad \text{where } w = f(z)$$

Pf: By the open mapping thm, f^{-1} is continuous and Ω is open. Since $z = f^{-1}(f(z))$ for each z, the result follows from the chain Rule.

8. Goursat's Thm

Recall "f is analytic" means f' exists and f' is continuous. In this Section, we will show "f' exists" $\Rightarrow$ "f is analytic"

Goursat's Thm. Let $G \subset \mathbb{C}$ open and let $f: G \to \mathbb{C}$ a differentiable function, then f is analytic on G

a differentiable function, then f is analytic

on G.

Pf: As the analyticity is local property, we can assume G is just an open disc.

We will show f is analytic by using Morera's Thm.

That is, we will show that $\int_T f = 0$ for each triangular path T in G.

Let $T = [a, b, c, a]$ and Δ be the closed set formed by T and its insider. Notice that $T = \partial\Delta$

Step1: Using midpts of the sides of Δ form four triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ inside Δ, and gives the bdry appropriate directions. Then each $T_j = \partial\Delta_j$ is a triangular path and

$$\int_T f = \sum_{j=1}^{4} \int_{T_j} f \qquad (1)$$

Among these four paths, there is one, call it $T^{(1)}$, such that $\left|\int_{T^{(1)}} f\right| \geq \left|\int_{T_j} f\right|$ for $j = 1, 2, 3, 4$.

Note the length of each T_j — denoted by $l(T_j)$ — equals $\frac{1}{2} l(T)$. Also $\operatorname{diam} T_j = \frac{1}{2} \operatorname{diam} T$;

finally, using (1),

finally, using (1),

$$\left|\int_T f\right| \le 4\left|\int_{T^{(1)}} f\right|$$

Repeating this process on $T^{(1)}$, and so on inductively. We get a sequence $\{T^{(n)}\}$ of closed triangular paths s.t if $\Delta^{(n)}$ is the inside of $T^{(n)}$ union $T^{(n)}$, then

$$\Delta^{(1)} \supset \Delta^{(2)} \supset \cdots;$$

$$\left|\int_{T^{(n)}} f\right| \le 4\left|\int_{T^{(n+1)}} f\right|;$$

$$l(T^{(n+1)}) = \frac{1}{2} l(T^{(n)});$$

$$\operatorname{diam} \Delta^{(n+1)} = \frac{1}{2} \operatorname{diam} \Delta^{(n)}$$

$\Rightarrow$

$$\left|\int_T f\right| \le 4^n \left|\int_{T^{(n)}} f\right| \qquad (2)$$

$$l(T^{(n)}) = \left(\frac{1}{2}\right)^n l \text{ where } l = l(T);$$

$$\operatorname{diam} \Delta^{(n)} = \left(\frac{1}{2}\right)^n d \text{ where } d = \operatorname{diam}(\Delta).$$

Since each $\Delta^{(n)}$ is closed, we have

Claim: $\bigcap_{n=1}^{\infty} \Delta^{(n)} = \{z_0\}$ for some $z_0 \in G$.

Pf of claim: Exercize (using chapter 2)

Fix $\varepsilon > 0$. Since f has a derivative at z_0 we can find a $\delta > 0$ s.t $B(z_0;\delta) \subset G$ and

$\left| f(z) - f(z_0) \right.$ ... $0 < |z - z_0| < \delta$

a $\delta > 0$ s.t. $B(z_0, \delta) \subset \Omega$ and

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

That is,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \varepsilon |z - z_0|$$

whenever $|z - z_0| < \delta$.

Choose n s.t diam $\Delta^{(n)} = (\frac{1}{2})^n d < \delta$. Since $z_0 \in \Delta^{(n)}$,
$\Rightarrow \Delta^{(n)} \subset B(z_0, \delta)$. Note Cauchy's Thm gives

$$0 = \int_{T^{(n)}} dz = \int_{T^{(n)}} z\, dz. \text{ Hence}$$

$$\left| \int_{T^{(n)}} f \right| = \left| \int_{T^{(n)}} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\, dz \right|$$

$$\le \varepsilon \int_{T^{(n)}} |z - z_0|\, |dz|$$

$$\le \varepsilon [\text{diam}\, \Delta^{(n)}] [l(T^{(n)})]$$

$$= \varepsilon\, d\, l\, (\tfrac{1}{4})^n$$

By (2), $\Rightarrow$

$$\left| \int_T f \right| \le 4^n \varepsilon\, d\, l\, (\tfrac{1}{4})^n = \varepsilon\, d\, l.$$

Since ε is arbitrary and d, l are fixed, we have

$$\int_T f = 0$$