2. (b) Calculate the cube roots of \( i \).

**Solution:** In polar coordinates, finding a cube root of \( i \) amounts to solving the equation

\[ r^3 e^{i3\theta} = e^{i\pi} \]

which forces

\[ r = 1, 3\theta = \frac{\pi}{2} + 2\pi k \]

with \( k \in \mathbb{Z} \). The possible values of \( \theta \) then must be \( \frac{\pi}{6} + \frac{2\pi k}{3} \), so \( \theta \) is one of \( \frac{\pi}{6}, \frac{5\pi}{6}, \) or \( \frac{3\pi}{2} \).

7. If \( z \in \mathbb{C} \) and \( \text{Re}(z^n) \geq 0 \) for every positive integer \( n \), show that \( z \) is a non-negative real number.

**Solution:** Again we put \( z \) into polar coordinates, \( z = re^{i\theta} \). Our assumption tells us that (up to an integer multiple of \( 2\pi \), at least)

\[ -\frac{\pi}{2} \leq n \cdot \theta \leq \frac{\pi}{2} \]

for every positive integer \( n \). This means \( \theta \) itself satisfies \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \). If \( |\theta| > 0 \), then we can take the largest \( n \) such that \( |n\theta| \leq \frac{\pi}{2} \). However,

\[
\begin{align*}
|\theta| &\leq |n\theta| + |\theta| \\
&\leq \frac{\pi}{2} + \frac{\pi}{2} \\
&= \pi
\end{align*}
\]

meaning \( \frac{\pi}{2} < |(n+1)\theta| \leq \pi \), which we’ve established to be impossible. Thus \( z \) is real, so \( z = \text{Re}(z) \geq 0 \).
Chapter 2.3

3. **Show that** \( \text{diam } A = \text{diam } A^-. \)

**Solution:** The left hand side is certainly \( \leq \) the right, since \( A^- \) contains \( A \) (so we are taking the sup over more points than we are for \( A \)'s diameter). For the other direction, by the definition of supremum as a least upper bound it’s enough to check that for each \( s \) and \( t \) in \( A \), \( d(s, t) \leq \text{diam } A \). We’ve seen any element in the closure is either in \( A \) or a limit point, so we can find a sequence \( s_i \in A \) with \( \lim_{n \to \infty} d(s, s_n) = 0 \). Likewise, we can take a sequence \( t_j \in A \) converging to \( t \). By the triangle inequality,

\[
d(s, t) - d(s, s_n) - d(t, t_n) \leq d(s, t) \leq d(s, t) + d(s, s_n) + d(t, t_n)
\]

so taking limits we see that \( \lim_{n \to \infty} d(s_n, t_n) = d(s, t) \). This shows that \( d(s, t) = \limsup_{n \to \infty} d(s_n, t_n) \) is certainly \( \leq \sup_{i,j} d(s_i, t_j) \), which in turn is \( \leq \sup_{x,y \in A} d(x, y) = \text{diam } A \) since all of the \( s_i \) and \( t_j \) are in \( A \), and we are done.

---

1. If \( s \) is actually in \( A \), then just take all the \( s_i \)'s to be \( s \)