Math 220A HW 1 Solutions to Selected Problems

Chapter 1.4

2. (b) **Calculate the cube roots of i.**

   Solution: In polar coordinates, finding a cube root of i amounts to solving the equation

$$r^3 e^{i3\theta} = e^{\frac{i\pi}{2}}$$

   which forces

$$r = 1, 3\theta = \frac{\pi}{2} + 2\pi k$$

   with $k \in \mathbb{Z}$. The possible values of θ then must be $\frac{\pi}{6} + \frac{2\pi k}{3}$, so θ is one of $\frac{\pi}{6}$, $\frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

7. **If $z \in \mathbb{C}$ and $\mathrm{Re}(z^n) \geq 0$ for every positive integer n, show that z is a non-negative real number.**

   Solution: Again we put z into polar coordinates, $z = re^{i\theta}$. Our assumption tells us that (up to an integer multiple of 2π, at least)

$$-\frac{\pi}{2} \leq n \cdot \theta \leq \frac{\pi}{2}$$

   for every positive integer n. This means θ itself satisfies $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. If $|\theta| > 0$, then we can take the largest n such that $|n\theta| \leq \frac{\pi}{2}$. However,

$$\begin{aligned}|(n+1)\theta| &\leq |n\theta| + |\theta| \\ &\leq \frac{\pi}{2} + \frac{\pi}{2} \\ &\leq \pi\end{aligned}$$

   meaning $\frac{\pi}{2} < |(n+1)\theta| \leq \pi$, which we've established to be impossible. Thus z is real, so $z = \mathrm{Re}(z) \geq 0$.

Chapter 2.3

3. **Show that** $\operatorname{diam} A = \operatorname{diam} A^-$.

Solution: The left hand side is certainly $\leq$ the right, since A^- contains A (so we are taking the sup over more points than we are for A's diameter). For the other direction, by the definition of supremum as a least upper bound it's enough to check that for each s and t in A, $d(s,t) \leq \operatorname{diam} A$. We've seen any element in the closure is either in A or a limit point, so we can find a sequence $s_i \in A$ with $\lim_{n\to\infty} d(s, s_n) = 0$ [1]. Likewise, we can take a sequence $t_j \in A$ converging to t. By the triangle inequality,

$$d(s,t) - d(s,s_n) - d(t,t_n) \leq d(s_n,t_n) \leq d(s,t) + d(s,s_n) + d(t,t_n)$$

so taking limits we see that $\lim_{n\to\infty} d(s_n,t_n) = d(s,t)$. This shows that $d(s,t) = \limsup_{n\to\infty} d(s_n,t_n)$ is certainly $\leq \sup_{i,j} d(s_i,t_j)$, which in turn is $\leq \sup_{x,y\in A} d(x,y) = \operatorname{diam} A$ since all of the s_i and t_j are in A, and we are done.

[1] if s is actually *in* A, then just take all the s_i's to be s