6. Describe the following sets: \{ z : e^z = i \}, \{ z : e^z = -1 \}, \{ z : e^z = -i \}, \\
\{ z : \cos z = 0 \}, \{ z : \sin z = 0 \}.

**Solution:** The general solution for $e^z = w$ is $z = \log|w| + i(\arg(w) + 2\pi k), k \in \mathbb{Z}$. Plugging in $w = -1, \pm i$, we find that

\[
\{ z : e^z = i \} = \{ i(\frac{1}{2} + 2k)\pi, k \in \mathbb{Z} \},
\]

\[
\{ z : e^z = -1 \} = \{ i((2k + 1)\pi, k \in \mathbb{Z} \},
\]

and

\[
\{ z : e^z = i \} = \{ i(2k - \frac{1}{2})\pi, k \in \mathbb{Z} \}
\]

For the rest, we can write $\cos z$ and $\sin z$ in terms of $e^{\pm iz}$ to see (using that $e^z$ is nonzero) that

\[
\{ z : \cos z = 0 \} = \{ z : e^{iz} = -e^{-iz} \}
\]

\[
= \{ z : e^{2iz} = -1 \}
\]

while

\[
\{ z : \sin z = 0 \} = \{ z : e^{iz} = e^{-iz} \}
\]

\[
= \{ z : e^{2iz} = 1 \}
\]

Applying our earlier results, we have

\[
\{ z : \cos z = 0 \} = \{ \frac{k}{2\pi}, k \text{ odd} \}, \{ z : \sin z = 0 \} = \{ k\pi, k \in \mathbb{Z} \}
\]

7. Prove formulas for $\cos(z + w)$ and $\sin(z + w)$.

**Solution:** As in the last homework, it’s helpful to use

\[
\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.
\]
With these, we can recover the usual sum formulas:

\[
\cos(z + w) = \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} = \frac{2e^{iz}e^{iw} + 2e^{-iz}e^{-iw}}{4} = \frac{2e^{iz}e^{iw} + 2e^{-iz}e^{-iw} + e^{-iz}e^{iw} + e^{iz}e^{-iw} + e^{iz}e^{iw} - e^{-iz}e^{-iw}}{4} = \frac{(e^{iz} + e^{-iz})(e^{iw} + e^{-iw})}{4} - \frac{(e^{iz} - e^{-iz})(e^{iw} - e^{-iw})}{(2i)^2} = \cos z \cos w - \sin z \sin w
\]

and

\[
\sin(z + w) = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \frac{2e^{iz}e^{iw} - 2e^{-iz}e^{-iw}}{4i} = \frac{2e^{iz}e^{iw} - 2e^{-iz}e^{-iw} + e^{-iz}e^{iw} - e^{-iz}e^{iw} + e^{iz}e^{-iw} - e^{iz}e^{-iw}}{4i} = \frac{(e^{iz} + e^{-iz})(e^{iw} + e^{-iw})}{4i} + \frac{(e^{iz} - e^{-iz})(e^{iw} - e^{-iw})}{4i} = \sin z \cos w + \cos z \sin w
\]

8. Define \( \tan z = \frac{\sin z}{\cos z} \). Where is this function defined and analytic?

**Solution:** By the quotient rule, \( \tan z \) is analytic on any open set where it is defined—ie, away from \( \cos z = 0 \). Explicitly, we know from the first problem that the domain of definition is (the open set)

\[ \mathbb{C} \setminus \left\{ \frac{k}{2} \pi, k \text{ odd} \right\} \]

12. Show that the real part of the function \( z^{\frac{1}{2}} \) is always positive.

**Solution:** The sole point here is that we have to choose a branch of the square root function: \( z^{\frac{1}{2}} \) is defined as \( e^{\frac{1}{2} \log z} \), where \( \log z \) is the principal branch of the logarithm, so we exclude non-negative real \( z \). Setting \( z = r^{i\theta} \) with \(-\pi < \theta < \pi\),

\[
z^{\frac{1}{2}} = e^{\frac{1}{2}(\log r + i\theta)} = r^{\frac{1}{2}} e^{\frac{\theta}{2}}
\]
Thus \( \text{Re} \, z^{\frac{1}{2}} = r^{\frac{1}{2}} \cos \left( \frac{\theta}{2} \right) \), which is positive as \( \frac{\theta}{2} \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \).

12. Suppose \( f : G \to \mathbb{C} \) is analytic and that \( G \) is connected. Show that if \( f(z) \) is real for all \( z \) in \( G \) then \( f \) is constant.

**Solution:** Since \( f \) is analytic on \( G \) \(^1\), we can write \( f = u + iv \) with \( u, v \) differentiable and satisfying the Cauchy Riemann equations. But \( v = 0 \) if \( f \) is real, so this amounts to

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0
\]

The derivation of the Cauchy Riemann equations also shows that \( f'(z) = \frac{\partial u}{\partial x}(z) - i \frac{\partial u}{\partial y}(z) \), so \( f'(z) = 0 \) in \( G \). As \( G \) is open and connected, we know that \( f'(z) = 0 \) implies \( f \) is constant on \( G \).

\(^1\)We should probably assume \( G \) is open—or at least contains an open set—here, otherwise we could take something like \( G = \) a real line segment and \( f = \) the identity, which is analytic on any open set containing \( G \). Since there’s some ambiguity, we might as well assume \( G \) is open.