Math 220A HW 3 Solutions

6. **Describe the following sets:** $\{z : \ e^z = i\}$, $\{z : \ e^z = -1\}$, $\{z : \ e^z = -i\}$, $\{z : \ \cos z = 0\}$, $\{z : \ \sin z = 0\}$.

   Solution: The general solution for $e^z = w$ is $z = \log|w| + i(\arg(w) + 2\pi k), k \in \mathbb{Z}$. Plugging in $w = -1, \pm i$, we find that

   $$\{z : \ e^z = i\} = \{i(\frac{1}{2} + 2k)\pi, k \in \mathbb{Z}\},$$

   $$\{z : \ e^z = -1\} = \{i((2k+1)\pi, k \in \mathbb{Z}\},$$

   and

   $$\{z : \ e^z = i\} = \{i(2k - \frac{1}{2})\pi, k \in \mathbb{Z}\}$$

   For the rest, we can write $\cos z$ and $\sin z$ in terms of $e^{\pm iz}$ to see (using that e^z is nonzero) that

   $$\begin{aligned}\{z : \ \cos z = 0\} &= \{z : \ e^{iz} = -e^{-iz}\} \\ &= \{z : \ e^{2iz} = -1\}\end{aligned}$$

   while

   $$\begin{aligned}\{z : \ \sin z = 0\} &= \{z : \ e^{iz} = e^{-iz}\} \\ &= \{z : \ e^{2iz} = 1\}\end{aligned}$$

   Applying our earlier results, we have

   $$\{z : \ \cos z = 0\} = \{\frac{k}{2}\pi, k \text{ odd}\}, \{z : \ \sin z = 0\} = \{k\pi, k \in \mathbb{Z}\}$$

7. **Prove formulas for** $\cos(z + w)$ **and** $\sin(z + w)$**.**

   Solution: As in the last homework, it's helpful to use

   $$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

With these, we can recover the usual sum formulas:

$$\begin{aligned}
\cos(z+w) &= \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} \\
&= \frac{2e^{iz}e^{iw} + 2e^{-iz}e^{-iw}}{4} \\
&= \frac{2e^{iz}e^{iw} + 2e^{-iz}e^{-iw} + e^{-iz}e^{iw} - e^{-iz}e^{iw} + e^{iz}e^{-iw} - e^{iz}e^{-iw}}{4} \\
&= \frac{e^{iz}e^{iw} + e^{-iz}e^{iw} + e^{iz}e^{-iw} + e^{-iz}e^{-iw} + e^{iz}e^{iw} - e^{-iz}e^{iw} - e^{iz}e^{-iw} + e^{-iz}e^{-iw}}{4} \\
&= \frac{(e^{iz} + e^{-iz})(e^{iw} + e^{-iw})}{4} - \frac{(e^{iz} - e^{-iz})(e^{iw} - e^{-iw})}{(2i)^2} \\
&= \cos z \cos w - \sin z \sin w
\end{aligned}$$

and

$$\begin{aligned}
\sin(z+w) &= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} \\
&= \frac{2e^{iz}e^{iw} - 2e^{-iz}e^{-iw}}{4i} \\
&= \frac{2e^{iz}e^{iw} - 2e^{-iz}e^{-iw} + e^{-iz}e^{iw} - e^{-iz}e^{iw} + e^{iz}e^{-iw} - e^{iz}e^{-iw}}{4i} \\
&= \frac{e^{iz}e^{iw} - e^{-iz}e^{iw} + e^{iz}e^{-iw} - e^{-iz}e^{-iw} + e^{iz}e^{iw} + e^{-iz}e^{iw} - e^{iz}e^{-iw} - e^{-iz}e^{-iw}}{4i} \\
&= \frac{(e^{iz} - e^{-iz})(e^{iw} + e^{-iw})}{4i} + \frac{(e^{iz} + e^{-iz})(e^{iw} - e^{-iw})}{4i} \\
&= \sin z \cos w + \cos z \sin w
\end{aligned}$$

8. **Define** $\tan z = \dfrac{\sin z}{\cos z}$**. Where is this function defined and analytic?**

   Solution: By the quotient rule, $\tan z$ is analytic on any open set where it is defined—ie, away from $\cos z = 0$. Explicitly, we know from the first problem that the domain of definition is (the open set)

$$\mathbb{C} \setminus \{\frac{k}{2}\pi, k \text{ odd}\}$$

12. **Show that the real part of the function** $z^{\frac{1}{2}}$ **is always positive.**

   Solution: The sole point here is that we have to choose a branch of the square root function: $z^{\frac{1}{2}}$ is defined as $e^{\frac{1}{2}\log z}$, where $\log z$ is the principal branch of the logarithm, so we exclude non-negative real z. Setting $z = r^{i\theta}$ with $-\pi < \theta < \pi$,

$$\begin{aligned}
z^{\frac{1}{2}} &= e^{\frac{1}{2}(\log r + i\theta)} \\
&= r^{\frac{1}{2}} e^{\frac{\theta}{2}}
\end{aligned}$$

Thus $\operatorname{Re} z^{\frac{1}{2}} = r^{\frac{1}{2}} \cos \frac{\theta}{2}$, which is positive as $\frac{\theta}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

12. **Suppose $f : G \to \mathbb{C}$ is analytic and that G is connected. Show that if $f(z)$ is real for all z in G then f is constant.**

Solution: Since f is analytic on G [1], we can write $f = u + iv$ with u, v differentiable and satisfying the Cauchy Riemann equations. But $v = 0$ if f is real, so this amounts to

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

The derivation of the Cauchy Riemann equations also shows that $f'(z) = \frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial y}(z)$, so $f'(z) = 0$ in G. As G is open and connected, we know that $f'(z) = 0$ implies f is constant on G.

[1] We should probably assume G is open—or at least contains an open set—here, otherwise we could take something like $G =$ a real line segment and $f =$ the identity, which is analytic on any open set containing G. Since there's some ambiguity, we might as well assume G is open.