6. Evaluate the following cross ratios:

(a) \((7 + i, 1, 0, \infty)\)

**Solution:** The formula for the cross ratio is

\[
(\omega_1, \omega_2, \omega_3, \omega_4) = \frac{\omega_1 - \omega_3 \omega_2 - \omega_4}{\omega_1 - \omega_4 \omega_2 - \omega_3}
\]

since putting \(\omega_2, \omega_3, \omega_4\) in the place of \(\omega_1\) has the desired properties. Since \(\omega_4 = \infty\) here, one can check that removing the terms involving \(\omega_4\) gives the correct images, and we have

\[
(7 + i, 1, 0, \infty) = \frac{(7 + i - 0)}{1 - 0} = 7 + i
\]

(b) \((2, 1 - i, 1, 1 + i)\)

**Solution:** This is even more straightforward:

\[
(2, 1 - i, 1, 1 + i) = \frac{1}{1 - i} \frac{-2i}{-i} = 1 + i
\]

(c) \((0, 1, i, -1)\)

**Solution:**

\[
(0, 1, i, -1) = \frac{-i}{1} \frac{2}{1 - i} = 1 - i
\]

(d) \((i - 1, \infty, 1 + i, 0)\)

**Solution:** By the same reasoning as in part a, we should remove the \(\omega_2\) terms here, yielding

\[
(i - 1, \infty, 1 + i, 0) = \frac{-2}{i - 1} = -(1 + i)
\]
7. If $Tz = \frac{az+b}{cz+d}$, find $z_2, z_3, z_4$ (in terms of $a, b, c, d$) such that $Tz = (z, z_2, z_3, z_4)$.

**Solution:** In other words, this is just solving

\[
Tz_2 = 1, Tz_3 = 0, Tz_4 = \infty.
\]

We’re assuming here that $T$ is a Möbius transformation (otherwise the cross ratio might be impossible to define!), so we must have

\[
T^{-1}1 = z_2, T^{-1}0 = z_3, T^{-1}\infty = z_4.
\]

Luckily, $T^{-1}$ isreadily computable in terms of $a, b, c, d$:

\[
T^{-1}z = \frac{dz - b}{-cz + a}.
\]

This shows that

\[
z_2 = \frac{d - b}{a - c}, z_3 = \frac{-b}{a}, z_4 = \frac{-d}{c}.
\]

8. If $Tz = \frac{az+b}{cz+d}$, show that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ iff we can choose $a, b, c, d$ to be real.

**Solution:** Again, we should assume $T$ is invertible, since otherwise it is constant and $T(\mathbb{R}_\infty)$ will certainly not be $\mathbb{R}_\infty$. The “if” direction is clear, so assume $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$.

We want to show that we can write $a, b, c, d$ as real multiples of a single element, so we look at their ratios. Plugging in 0 and $\infty$, we get that $\frac{a}{c}$ and $\frac{b}{d}$ are real as long as the denominators are nonzero. However, $T$ is invertible, so $T^{-1}(\mathbb{R}_\infty) = \mathbb{R}_\infty$, and the same procedure shows that $\frac{d}{c}, \frac{b}{a}$ are real with the same caveats.

Notice also that if we compose with any other Möbius transformation preserving $\mathbb{R}_\infty$, the resulting transformation $S$ will still preserve $\mathbb{R}_\infty$. Doing so with $z \mapsto \frac{1}{z}$, we get

\[
Sz = \frac{cz + d}{az + b}
\]

which must also preserve $\mathbb{R}_\infty$. This means that (unless $a = d = 0$, which we will cover later) we can assume that $cd \neq 0$, since otherwise we can swap them with $a$ and $b$, and these are guaranteed to be nonzero if $c$ or $d$ (respectively) is by the invertibility of $T$.

This being the case, the way forward is clear: $\frac{a}{c}$ and $\frac{d}{c}$ are already in $\mathbb{R}$, and so is $\frac{b}{c} = \frac{b}{d} \frac{d}{c}$. This is true in the only case we didn’t cover as well, because then $Tz = \frac{b}{cz}$, so $T(1)$ being real gives us what we want. Therefore, we can scale everything by $c^{-1}$ to get real numbers.
9. If \( Tz = \frac{az + b}{cz + d} \), find necessary and sufficient conditions that \( T(\Gamma) = \Gamma \).

Solution: As before, \( T^{-1} \) will also preserve \( \Gamma \) if and only if \( T \) does, and therefore it will preserve symmetry with respect to \( \Gamma \). In particular, 0 and \( \infty \) will be sent to a symmetric pair. Let \( a = T^{-1}(0) \), and note that \( a \) cannot be in \( \Gamma \) by our assumption. We can check (for example, using the calculation on page 51 of the book) that \( a \)'s inverse with respect to \( \Gamma \) is \( a^* := \frac{1}{\bar{a}} \). This means that \( T \) must be of the form

\[
Tz = b\frac{z - a}{z - a^*}, b \in \mathbb{C}^\times.
\]

If \( a = 0 \) or \( \infty \), then \( Tz = bz \) or \( \frac{b}{z} \), and so \( |b| = 1 \) is the only condition needed for \( T(\Gamma) = \Gamma \). Otherwise, we can pull out an \( \bar{a} \) from the bottom to rewrite \( T \) in the form

\[
Tz = cz - a\frac{z - a^*}{\bar{a} - 1}.
\]

Which \( c \) are allowed? We can find out by observing that

\[
|Tz|^2 = \frac{(z - a)(\bar{z} - \bar{a})}{(\bar{z} - 1)(\bar{a}z - 1)}
\]

\[
= |c|^2 \frac{|a|^2 + |z|^2 - (\bar{a}z + \bar{z}a)}{|a|^2 |z|^2 + 1 - (\bar{a}z + \bar{z}a)}.
\]

If \( z \in \Gamma \), the right hand side reduces to \( |c|^2 \), meaning \( T(\Gamma) = \Gamma \) if and only if \( c \in \Gamma \). Thus, we can say that \( T(\Gamma) = \Gamma \) if and only if

\[
T(z) = c\frac{z - a}{\bar{a}z - 1}, c \in \Gamma, a \notin \Gamma
\]

with the usual convention if \( a = \infty \).

10. Let \( D = \{ z : |z| < 1 \} \), and find all Möbius transformations \( T \) such that \( T(D) = D \).

Solution: By continuity, any such \( T \) satisfy \( T(\Gamma) = \Gamma \) as well, so we need only check among the \( T \) described in previous problem. So let

\[
Tz = c\frac{z - a}{\bar{a}z - 1}, c \in \Gamma
\]

Since \( Ta = 0 \in D \), we must have \( a \in D \) as well. We claim this is the only condition necessary. To see this, we just need to check whether \( |Tz|^2 < 1 \) for \( z \in D \). By our earlier calculation, this is equivalent to asking whether

\[
|a|^2 + |z|^2 - (\bar{a}z + \bar{z}a) < |a|^2 |z|^2 + 1 - (\bar{a}z + \bar{z}a),
\]

or in other words, whether

\[
|a|^2 + |z|^2 < |a|^2 |z|^2 + 1.
\]

This is true whenever \( |z| < 1 \), so we are done.
the denominator in that expression is never 0 if $\bar{a}, z \in D$