Math 220A HW 4 Solutions

6. **Evaluate the following cross ratios:**

   (a) $(7+i, 1, 0, \infty)$

   Solution: The formula for the cross ratio is

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \frac{z_2 - z_4}{z_2 - z_3}$$

   since putting z_2, z_3, z_4 in the place of z_1 has the desired properties. Since $z_4 = \infty$ here, one can check that removing the terms involving z_4 gives the correct images, and we have

$$\begin{aligned}(7+i, 1, 0, \infty) &= \frac{(7+i-0)}{1-0} \\ &= 7+i\end{aligned}$$

   (b) $(2, 1-i, 1, 1+i)$

   Solution: This is even more straightforward:

$$\begin{aligned}(2, 1-i, 1, 1+i) &= \frac{1}{1-i} \frac{-2i}{-i} \\ &= 1+i\end{aligned}$$

   (c) $(0, 1, i, -1)$

   Solution:

$$\begin{aligned}(0, 1, i, -1) &= \frac{-i}{1} \frac{2}{1-i} \\ &= 1-i\end{aligned}$$

   (d) $(i-1, \infty, 1+i, 0)$

   Solution: By the same reasoning as in part a, we should remove the z_2 terms here, yielding

$$\begin{aligned}(i-1, \infty, 1+i, 0) &= \frac{-2}{i-1} \\ &= -(1+i)\end{aligned}$$

7. **If $Tz = \frac{az+b}{cz+d}$, find z_2, z_3, z_4 (in terms of a, b, c, d) such that $Tz = (z, z_2, z_3, z_4)$.**

   Solution: In other words, this is just solving

   $$Tz_2 = 1, Tz_3 = 0, Tz_4 = \infty.$$

   We're assuming here that T is a Möbius transformation (otherwise the cross ratio might be impossible to define!), so we must have

   $$T^{-1}1 = z_2, T^{-1}0 = z_3, T^{-1}\infty = z_4.$$

   Luckily, T^{-1} is readibly computable in terms of a, b, c, d:

   $$T^{-1}z = \frac{dz - b}{-cz + a}.$$

   This shows that

   $$z_2 = \frac{d-b}{a-c}, z_3 = -\frac{b}{a}, z_4 = -\frac{d}{c}.$$

8. **If $Tz = \frac{az+b}{cz+d}$, show that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ iff we can choose a, b, c, d to be real.**

   Solution: Again, we should assume T is invertible, since otherwise it is constant and $T(\mathbb{R}_\infty)$ will certainly not be $\mathbb{R}_\infty$. The "if" direction is clear, so assume $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. We want to show that we can write a, b, c, d as real multiples of a single element, so we look at their ratios. Plugging in 0 and ∞, we get that $\frac{a}{c}$ and $\frac{b}{d}$ are real as long as the denominators are nonzero. However, T is invertible, so $T^{-1}(\mathbb{R}_\infty) = \mathbb{R}_\infty$, and the same procedure shows that $\frac{d}{c}, \frac{b}{a}$ are real with the same caveats.

   Notice also that if we compose with any other Möbius transformation preserving $\mathbb{R}_\infty$, the resulting transformation S will still preserve $\mathbb{R}_\infty$. Doing so with $z \mapsto \frac{1}{z}$, we get

   $$Sz = \frac{cz + d}{az + b}$$

   which must also preserve $\mathbb{R}_\infty$. This means that (unless $a = d = 0$, which we will cover later) we can assume that $cd \neq 0$, since otherwise we can swap them with a and b, and these are guaranteed to be nonzero if c or d (respectively) is by the invertibility of T.

   This being the case, the way forward is clear: $\frac{a}{c}$ and $\frac{d}{c}$ are already in $\mathbb{R}$, and so is $\frac{b}{c} = \frac{b}{d}\frac{d}{c}$. This is true in the only case we didn't cover as well, because then $Tz = \frac{b}{cz}$, so $T(1)$ being real gives us what we want. Therefore, we can scale everything by c^{-1} to get real numbers.

9. **If $Tz = \frac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$.**

   Solution: As before, T^{-1} will also preserve Γ if and only if T does, and therefore it will preserve symmetry with respect to Γ. In particular, 0 and ∞ will be sent to a symmetric pair. Let $a = T^{-1}(0)$, and note that a cannot be in Γ by our assumption. We can check (for example, using the calculation on page 51 of the book) that a's inverse with respect to Γ is $a^* := \frac{1}{\bar{a}}$. This means that T must be of the form

   $$Tz = b\frac{z-a}{z-a^*}, b \in \mathbb{C}^\times.$$

   If $a = 0$ or ∞, then $Tz = bz$ or $\frac{b}{z}$, and so $|b| = 1$ is the only condition needed for $T(\Gamma) = \Gamma$. Otherwise, we can pull out an $\bar{a}$ from the bottom to rewrite T in the form

   $$Tz = c\frac{z-a}{\bar{a}z-1}.$$

   Which c are allowed? We can find out by observing that

   $$\begin{aligned}|Tz|^2 &= \frac{(z-a)(\bar{z}-\bar{a})}{(\bar{a}z-1)(a\bar{z}-1)} \\ &= |c|^2\frac{|a|^2+|z|^2-(\bar{a}z+\bar{z}a)}{|a|^2|z|^2+1-(\bar{a}z+\bar{z}a)}.\end{aligned}$$

   If $z \in \Gamma$, the right hand side reduces to $|c|^2$, meaning $T(\Gamma) = \Gamma$ if and only if $c \in \Gamma$. Thus, we can say that $T(\Gamma) = \Gamma$ if and only if

   $$T(z) = c\frac{z-a}{\bar{a}z-1}, c \in \Gamma, a \notin \Gamma$$

   with the usual convention if $a = \infty$.

10. **Let $D = \{z : |z| < 1\}$, and find all Möbius transformations T such that $T(D) = D$.**

    Solution: By continuity, any such T satisfy $T(\Gamma) = \Gamma$ as well, so we need only check among the T described in previous problem. So let

    $$Tz = c\frac{z-a}{\bar{a}z-1}, c \in \Gamma$$

    Since $Ta = 0 \in D$, we must have $a \in D$ as well. We claim this is the only condition necessary. To see this, we just need to check whether $|Tz|^2 < 1$ for $z \in D$. By our earlier calculation, this is equivalent [1] to asking whether

    $$|a|^2 + |z|^2 - (\bar{a}z + \bar{z}a) < |a|^2|z|^2 + 1 - (\bar{a}z + \bar{z}a),$$

    or in other words, whether

    $$|a|^2 + |z|^2 < |a|^2|z|^2 + 1.$$

    This is true whenever $|z| < 1$, so we are done.

[1] the denominator in that expression is never 0 if $\bar{a}, z \in D$