Math 220A HW 5 Solutions

21. **Let T be a Möbius transformation with fixed points z_1 and z_2. If S is a Móbius transformation show that $S^{-1}TS$ has fixed points $S^{-1}z_1$ and $S^{-1}z_2$.**

    Solution: This can be computed directly:

    $$\begin{aligned} S^{-1}TS(S^{-1}z_i) &= S^{-1}T(z_i) \\ &= S^{-1}z_i \end{aligned}$$

22. (a) **Show that a Möbius transformation has 0 and ∞ as its only fixed points iff it is a dilation, but not the identity.**

    Solution: Conflicting with the definition from class, a dilation here is scaling by any $\lambda \in \mathbb{C}^\times$. It is easy that a nonidentity dilation has its only fixed points $0, \infty$. On the other hand, let $Tz = \frac{az+b}{cz+d}$. Then $T(\infty) = \infty$ implies $c = 0$, while $0 = T(0) = \frac{b}{d}$ forces $b = 0$ as well. Thus T is a dilation, non identity since it only has these two fixed points.

    (b) **Show that a Möbius transformation has ∞ as its only fixed point iff it is a translation, but not the identity.**

    Solution: Similarly, a nonidentity translation certainly has ∞ as its only fixed point. The calculation in the previous part shows that we can write $Tz = az + b$, $a \neq 0$. If $a \neq 1$, then $az + b = z$ has (non-infinite) solution $z = \frac{b}{a-1}$, and therefore a finite fixed point. Thus $a = 1$, so T is a translation, non identity since it only has one fixed point.

24. **Let T be a Möbius transformation, $T \neq$ the identity. Show that a Möbius transformation S commutes with T if S and T have the same fixed points. (Hint: Use Exercises 21 and 22.)**

    Solution: T (and S, since it has the same fixed points as T) is not the identity, so we can let z_1, z_2 be its fixed points. These are also the fixed points of S^{-1}. Möbius transformations are thrice transitive, so we can certainly find a Möbius transformation U that sends z_1 to 0 and z_2 to ∞, or z_1 to ∞ if $z_1 = z_2$. By problem 21, the transformations

    $$UTU^{-1}, USU^{-1}, US^{-1}U^{-1}$$

    all have fixed points $Uz_1 = 0, Uz_2 = \infty$ in the first case, and $Uz_1 = Uz_2 = \infty$ in the second [1]. Now we can apply problem 22 to conclude that they are either all dilations or all translations. But any two dilations commute, and the same is true for translations, so we know either way that

$$\begin{aligned} UTU^{-1} &= (USU^{-1})(UTU^{-1})(US^{-1}U^{-1}) \\ &= USTS^{-1}U^{-1} \end{aligned}$$

Conjugating by U^{-1} yields

$$T = STS^{-1},$$

which means S and T commute.

[1] in the second case, if w was another fixed point of UTU^{-1}, then $U^{-1}w \neq z_1$ would be a second fixed point of T