Math 220A HW 6 Solutions

Section 4.1

12. **Let** $I(r) = \int_\gamma \frac{e^{iz}}{z} dz$ **where** $\gamma : [0, \pi] \to \mathbb{C}$ **is defined by** $\gamma(t) = re^{it}$**. Show that** $\lim_{r\to\infty} I(r) = 0$**.**

Solution: We'll use the bound $|\int_\gamma f| \leq \int_\gamma |f||dz|$, for $f(z) = \frac{e^{iz}}{z}$. Since $|\gamma|(s) = V(\gamma; [0, s]) = \int_0^s |re^{it}|dt = rs$,

$$
\begin{aligned}
|I(r)| &\leq \int_\gamma |f||dz| \\
&= \int_0^\pi \left|\frac{e^{rie^{it}}}{re^{it}}\right| r\,dt \\
&= \int_0^\pi |e^{r(i\cos(t)-\sin(t)}|dt \\
&= \int_0^\pi e^{-r\sin t}dt,
\end{aligned}
$$

and we are looking for a bound for this latter integral. If we could replace $\sin t$ with some multiple of t, the integral would be easy to evaluate, so we would like $e^{-r\sin t} \leq e^{-r\alpha t}$; ie, $\alpha t \leq \sin t$ on $[0, \pi]$, $\alpha \in \mathbb{R}_{>0}$. If we let $g(t) := \sin t - \alpha t$, we are looking for $g(t) \geq 0$.

Observe first that $g(0) = 0$ for every positive α, and the derivative $g'(t) = \cos t - \alpha$ will at first be positive if $\alpha < 1$. Since $g(\pi) < 0$, by continuity g will attain another zero—c_α, say—on $[0, \pi]$. This is not ideal, since the bound we want will not hold on $(c_\alpha, \pi]$. Luckily, $\sin t$ is symmetric about $\frac{\pi}{2}$ in this interval, so

$$\int_0^\pi e^{-r\sin t}dt = 2\int_0^{\frac{\pi}{2}} e^{-r\sin t}dt$$

meaning we only need the bound to hold on $[0, \frac{\pi}{2}]$. In other words, we would like $g(\frac{\pi}{2}) = 1 - \frac{\pi}{2}\alpha \geq 0$, and for there to be no other zeros of g on $[0, \frac{\pi}{2}]$. In fact, the first condition implies the second. To see this, note that $g'(t) \geq 0$ on $[0, \arccos\alpha]$ and is negative on $(\arccos\alpha, \frac{\pi}{2})$ [1]. If there was another zero t_0 inside the interval, g would have to be increasing somewhere on $(t_0, \frac{\pi}{2})$ (otherwise $g(t_0)$ would be $> g(\frac{\pi}{2})$). But then by our discussion of the sign of g', we would have g increasing on all of $(0, t_0)$, which would force $g(t_0) > g(0)$. Thus, we can take any α with $0 < \alpha \leq \frac{2}{\pi}$; we may as well take $\alpha = \frac{2}{\pi}$. Finally we have that

$$\begin{aligned} I(r) &\leq 2 \int_0^{\frac{\pi}{2}} e^{-r\frac{2}{\pi}t} dt \\ &= -\frac{2\pi}{2r}(e^{-r} - 1). \end{aligned}$$

This certainly goes to 0 as r approaches ∞, so

$$\lim_{r \to \infty} I(r) = 0.$$

13. **Find $\int_\gamma z^{-\frac{1}{2}} dz$ where:**

(a) **γ is the upper half of the unit circle from $+1$ to -1:**

Solution: Since we're implicitly using the principal branch of the log, we should parametrize γ as $\gamma(t) = e^{it}, t \in [0, \pi]$. This allows us to calculate

$$\begin{aligned} \int_\gamma z^{-\frac{1}{2}} dz &= \int_0^{\pi} e^{-i\frac{t}{2}} i e^{it} dt \\ &= \int_0^{\pi} i e^{i\frac{t}{2}} dt \\ &= 2(e^{i\frac{\pi}{2}} - e^0) \\ &= 2(i - 1). \end{aligned}$$

Since $z^{-\frac{1}{2}}$ is not actually defined at -1, we probably should be taking the limit of the integral from 0 to s as s approaches π, but we will get the same result.

(b) **γ is the lower half of the unit circle from $+1$ to -1.**

Solution: Similarly, we find that

$$\begin{aligned} \int_\gamma z^{-\frac{1}{2}} dz &= \int_0^{-\pi} e^{-i\frac{t}{2}} i e^{it} dt \\ &= 2(e^{-i\frac{\pi}{2}} - e^0) \end{aligned}$$

[1] we know $\arccos\alpha$ will be in this interval because, for example, $g'(\frac{\pi}{2}) = -\alpha < 0$, so $\cos t = \alpha$ has a solution there.

$$= 2(-i-1)$$

19. **Let** $\gamma(t) = 1 + e^{it}$ **for** $0 \le t \le 2\pi$ **and find** $\int_\gamma (z^2-1)^{-1} dz$**.**

    Solution: We rewrite $\frac{1}{z^2-1}$ as

$$\begin{aligned}\frac{1}{z^2-1} &= \frac{1}{2}\frac{z+1-(z-1)}{z^2-1} \\ &= \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)\end{aligned}$$

    in order to calculate (noting that $\log(2+e^{it})$ is defined on an open set containing $[0, 2\pi]$)

$$\begin{aligned}\int_\gamma \frac{1}{z^2-1} dz &= \int_0^{2\pi} i \frac{e^{it}}{(2+e^{it})e^{it}} dt \\ &= \int_0^{2\pi} \frac{1}{2}\left(\frac{1}{e^{it}} - \frac{1}{2+e^{it}}\right) ie^{it} dt \\ &= \int_0^{2\pi} \frac{1}{2}\left(i - \frac{ie^{it}}{2+e^{it}}\right) dt \\ &= \frac{1}{2}(it - \log(2+e^{it}))\Big|_0^{2\pi} \\ &= \frac{1}{2}(2\pi i - \log(3) + \log(3)) \\ &= \pi i.\end{aligned}$$

20. **Let** $\gamma(t) = 2e^{it}$ **for** $-\pi \le t \le \pi$ **and find** $\int_\gamma (z^2-1)^{-1} dz$**.**

    Solution: Notice that $Tz = \frac{z-1}{z+1}$ preserves $\mathbb{R}_\infty$, and therefore Tz is in $\mathbb{R}_{\le 0} \cup \{\infty\}$ if and only if z is real and $-1 \le z \le 1$. This means that away from that interval—and in particular, on an open set containing γ—$\log \frac{z-1}{z+1}$ is defined. However, we can deduce from the calculations in the previous problem that $\frac{1}{2} \log \frac{z-1}{z+1}$ is a primitive of $f(z) := \frac{1}{z^2-1}$, so the integral of f around the closed curve γ will be 0.

Section 4.2

7. **Use the results of this section to evaluate the following integrals:**

(d)

$$\int_\gamma \frac{\log z}{z^n} dz, \gamma(t) = 1 + \frac{1}{2}e^{it}, 0 \leq t \leq 2\pi \textbf{ and } n \geq 0.$$

Solution: Both $\log z$ and z^n are analytic, and z^n has no zeroes on $B(1; \frac{1}{2} + \epsilon)$ for small ϵ, so $\frac{\log z}{z^n}$ is analytic there. Thus, its integral around a closed curve is 0.

9. **Evaluate the following integrals:**

(a)

$$\int_\gamma \frac{e^z - e^{-z}}{z^n} dz, \textbf{ where } n \textbf{ is any positive integer and } \gamma(t) = e^{it}, 0 \leq t \leq 2\pi.$$

Solution: If $f(z) = e^z - e^{-z}$, then we know that

$$\int_\gamma \frac{f(z)}{z^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0).$$

Of course, $f^(n-1)(z) = e^z + (-1)^{n-1}e^{-z}$, so we can write this as

$$\int_\gamma \frac{e^z - e^{-z}}{z^n} dz = \begin{cases} \frac{4\pi i}{(n-1)!}, n \text{ even} \\ 0, \ n \text{ odd} \end{cases}$$

(e)

$$\int_\gamma \frac{z^{\frac{1}{m}}}{(z-1)^m} dz, \textbf{ where } \gamma(t) = 1 + \frac{1}{2}e^{it}, 0 \leq t \leq 2\pi$$

Solution: If $m \leq 0$, then $\frac{z^{\frac{1}{m}}}{(z-1)^m}$ is analytic on $B(1; \frac{1}{2}+\epsilon)$, so its integral is 0 around γ. For m positive, $f(z) := z^{\frac{1}{m}}$ at least is analytic there, and we can again use that

$$\int_\gamma \frac{f(z)}{(z-1)^m} dz = \frac{2\pi i}{(m-1)!} f^{m-1}(1).$$

We can also check that

$$\begin{aligned} f^{(m-1)}(z) &= \frac{1}{m}(\frac{1}{m} - 1) \cdots (\frac{1}{m} - (m-2)) z^{\frac{1}{m} - (m-1)} \\ &= \frac{1}{m}(\frac{1-m}{m}) \cdots (\frac{1-(m-2)m}{m}) z^{\frac{1}{m} - (m-1)} \end{aligned}$$

$$= \frac{\prod_{j=0}^{m-2}(1-jm)}{m^{m-1}} z^{\frac{1}{m}-(m-1)}$$

(this works even for $m = 1$), and we arrive at

$$\int_{\gamma} \frac{f(z)}{(z-1)^m} dz = \frac{2\pi i \prod_{j=0}^{m-2}(1-jm)}{m^{m-1}(m-1)!}.$$

10. **Evaluate** $\int_{\gamma} \frac{z^2+1}{z(z^2+4)} dz$**, where** $\gamma(t) = re^{it}, 0 \le t \le 2\pi$**, for all possible values of** r**,** $0 < r < 2$**,** $2 < r < \infty$**.**

Solution: Assume first that $0 < r < 2$; then $f(z) := \frac{z^2+1}{z^2+4}$ is analytic on an open ball around 0 containing γ, so the integral is just $2\pi i \cdot f(0) = \frac{i\pi}{2}$.

Now suppose $2 < r < \infty$. Then the function has 3 poles in any ball around 0 containing γ, so we should use the partial fraction decomposition

$$\begin{aligned} \frac{z^2+1}{z(z^2+4)} &= \frac{1}{4z} + \frac{3z}{4(z^2+4)} \\ &= \frac{1}{4z} + \frac{3}{8}\left(\frac{1}{z+2i} + \frac{1}{z-2i}\right) \end{aligned}$$

to integrate term-by-term. Each of these terms is easy: since $\pm 2i$ are both at distance $2 < r$ from 0, we can apply Proposition 2.6 to yield

$$\begin{aligned} \int_{\gamma} \frac{1}{4z} + \frac{3}{8}\left(\frac{1}{z+2i} + \frac{1}{z-2i}\right) dz &= \frac{1}{4}\int_{\gamma} \frac{1}{z} dz + \frac{3}{8}\left(\int_{\gamma} \frac{1}{z+2i} dz + \int_{\gamma} \frac{1}{z-2i} dz\right) \\ &= \left(\frac{1}{4} + \frac{3}{8} + \frac{3}{8}\right) 2\pi i \\ &= 2\pi i. \end{aligned}$$