Math 220A HW 6 Solutions

Section 4.1

12. Let \( I(r) = \int_{\gamma} \frac{e^{iz}}{z} \, dz \) where \( \gamma : [0, \pi] \to \mathbb{C} \) is defined by \( \gamma(t) = re^{it} \). Show that
\[
\lim_{r \to \infty} I(r) = 0.
\]

Solution: We’ll use the bound \(|\int_{\gamma} f| \leq \int_{\gamma} |f||dz|\), for \( f(z) = \frac{e^{iz}}{z} \). Since \(|\gamma(s) = V(\gamma; [0, s]) = \int_{0}^{s} |re^{it}| \, dt = rs|,

\[
|I(r)| \leq \int_{\gamma} |f||dz|
= \int_{0}^{\pi} \left| \frac{e^{rie^{it}}}{re^{it}} \right| r \, dt
= \int_{0}^{\pi} |e^{ri(cos(t) - \sin(t))}| \, dt
= \int_{0}^{\pi} e^{-r \sin t} \, dt,
\]

and we are looking for a bound for this latter integral. If we could replace \( \sin t \) with some multiple of \( t \), the integral would be easy to evaluate, so we would like \( e^{-r \sin t} \leq e^{-r \alpha t} \); ie, \( \alpha t \leq \sin t \) on \([0, \pi] \), \( \alpha \in \mathbb{R}_{>0} \). If we let \( g(t) := \sin t - \alpha t \), we are looking for \( g(t) \geq 0 \).

Observe first that \( g(0) = 0 \) for every positive \( \alpha \), and the derivative \( g'(t) = \cos t - \alpha \) will at first be positive if \( \alpha < 1 \). Since \( g(\pi) < 0 \), by continuity \( g \) will attain another zero—\( c_\alpha \), say—on \([0, \pi] \). This is not ideal, since the bound we want will not hold on \((c_\alpha, \pi]\). Luckily, \( \sin t \) is symmetric about \( \frac{\pi}{2} \) in this interval, so

\[
\int_{0}^{\pi} e^{-r \sin t} \, dt = 2 \int_{0}^{\pi/2} e^{-r \sin t} \, dt
\]

meaning we only need the bound to hold on \([0, \frac{\pi}{2}] \). In other words, we would like \( g(\frac{\pi}{2}) = 1 - \frac{\pi}{2} \alpha \geq 0 \), and for there to be no other zeros of \( g \) on \([0, \frac{\pi}{2}] \). In fact, the first
condition implies the second. To see this, note that $g'(t) \geq 0$ on $[0, \arccos \alpha]$ and is negative on $(\arccos \alpha, \pi/2)$. If there was another zero $t_0$ inside the interval, $g$ would have to be increasing somewhere on $(t_0, \pi/2)$ (otherwise $g(t_0)$ would be $> g(\pi/2)$). But then by our discussion of the sign of $g'$, we would have $g$ increasing on all of $(0, t_0)$, which would force $g(t_0) > g(0)$. Thus, we can take any $\alpha$ with $0 < \alpha \leq \pi/2$; we may as well take $\alpha = \pi/2$.

Finally we have that

$$I(r) \leq 2 \int_0^{\pi/2} e^{-r \frac{2}{\pi} t} dt$$

$$= -\frac{2\pi}{2r} (e^{-r} - 1).$$

This certainly goes to 0 as $r$ approaches $\infty$, so

$$\lim_{r \to \infty} I(r) = 0.$$

13. Find $\int_{\gamma} z^{-\frac{1}{2}} dz$ where:

(a) $\gamma$ is the upper half of the unit circle from $+1$ to $-1$:

Solution: Since we’re implicitly using the principal branch of the log, we should parametrize $\gamma$ as $\gamma(t) = e^{it}, t \in [0, \pi]$. This allows us to calculate

$$\int_{\gamma} z^{-\frac{1}{2}} dz = \int_0^\pi e^{-i\frac{1}{2}i}e^{it} dt$$

$$= \int_0^\pi ie^{i\frac{1}{2}t} dt$$

$$= 2(e^{\pi i/2} - e^0)$$

$$= 2(i - 1).$$

Since $z^{-\frac{1}{2}}$ is not actually defined at $-1$, we probably should be taking the limit of the integral from 0 to $s$ as $s$ approaches $\pi$, but we will get the same result.

(b) $\gamma$ is the lower half of the unit circle from $+1$ to $-1$.

Solution: Similarly, we find that

$$\int_{\gamma} z^{-\frac{1}{2}} dz = \int_0^{-\pi} e^{-i\frac{1}{2}i}e^{it} dt$$

$$= 2(e^{-\pi i/2} - e^0)$$

1 we know $\arccos \alpha$ will be in this interval because, for example, $g'(\pi/2) = -\alpha < 0$, so $\cos t = \alpha$ has a solution there.
19. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$ and find $\int_\gamma (z^2 - 1)^{-1}dz$.

**Solution:** We rewrite $\frac{1}{z^2 - 1}$ as

$$\frac{1}{z^2 - 1} = \frac{1}{2} \left( \frac{1}{z - 1} - \frac{1}{z + 1} \right)$$

in order to calculate (noting that $\log(2 + e^{it})$ is defined on an open set containing $[0, 2\pi]$)

$$\int_\gamma \frac{1}{z^2 - 1} \, dz = \int_0^{2\pi} \frac{i}{2} \left( \frac{1}{e^{it}} - \frac{1}{2 + e^{it}} \right) e^{it} \, dt$$

$$= \int_0^{2\pi} \left( \frac{i}{2} \left( 1 - \frac{1}{2 + e^{it}} \right) \right) dt$$

$$= \int_0^{2\pi} \frac{i}{2} (e^{it} - 2) \, dt$$

$$= \frac{i}{2} \left( i t - \log(2 + e^{it}) \right) \bigg|_0^{2\pi}$$

$$= \frac{1}{2} (2\pi i - \log(3) + \log(3))$$

$$= \pi i.$$

20. Let $\gamma(t) = 2e^{it}$ for $-\pi \leq t \leq \pi$ and find $\int_\gamma (z^2 - 1)^{-1}dz$.

**Solution:** Notice that $Tz = \frac{z - 1}{z + 1}$ preserves $\mathbb{R}_\infty$, and therefore $Tz$ is in $\mathbb{R}_{\leq 0} \cup \{\infty\}$ if and only if $z$ is real and $-1 \leq z \leq 1$. This means that away from that interval—and in particular, on an open set containing $\gamma$—$\log \frac{z - 1}{z + 1}$ is defined. However, we can deduce from the calculations in the previous problem that $\frac{1}{2} \log \frac{z - 1}{z + 1}$ is a primitive of $f(z) := \frac{1}{z^2 - 1}$, so the integral of $f$ around the closed curve $\gamma$ will be 0.
Section 4.2

7. Use the results of this section to evaluate the following integrals:

(d) \[ \int_{\gamma} \frac{\log z}{z^n} dz, \quad \gamma(t) = 1 + \frac{1}{2} e^{it}, \quad 0 \leq t \leq 2\pi \text{ and } n \geq 0. \]

Solution: Both \( \log z \) and \( z^n \) are analytic, and \( z^n \) has no zeroes on \( B(1; \frac{1}{2} + \epsilon) \) for small \( \epsilon \), so \( \frac{\log z}{z^n} \) is analytic there. Thus, its integral around a closed curve is 0.

9. Evaluate the following integrals:

(a) \[ \int_{\gamma} \frac{e^z - e^{-z}}{z^n} dz, \quad \text{where } n \text{ is any positive integer and } \gamma(t) = e^{it}, \quad 0 \leq t \leq 2\pi. \]

Solution: If \( f(z) = e^z - e^{-z} \), then we know that

\[ \int_{\gamma} \frac{f(z)}{z^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0). \]

Of course, \( f^{(n-1)}(z) = e^z + (-1)^{n-1} e^{-z} \), so we can write this as

\[ \int_{\gamma} \frac{e^z - e^{-z}}{z^n} dz = \begin{cases} 4\pi i \frac{1}{(n-1)!}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \]

(e) \[ \int_{\gamma} \frac{z^{\frac{1}{m}}}{(z-1)^m} dz, \quad \text{where } \gamma(t) = 1 + \frac{1}{2} e^{it}, \quad 0 \leq t \leq 2\pi \]

Solution: If \( m \leq 0 \), then \( \frac{z^{\frac{1}{m}}}{(z-1)^m} \) is analytic on \( B(1; \frac{1}{2} + \epsilon) \), so its integral is 0 around \( \gamma \). For \( m \) positive, \( f(z) := z^{\frac{1}{m}} \) at least is analytic there, and we can again use that

\[ \int_{\gamma} \frac{f(z)}{(z-1)^m} dz = \frac{2\pi i}{(m-1)!} f^{m-1}(1). \]

We can also check that

\[ f^{(m-1)}(z) = \frac{1}{m} \left( \frac{1}{m} - 1 \right) \cdots \left( \frac{1}{m} - (m-2) \right) z^{\frac{1}{m}-(m-1)} \]
\[ = \frac{1}{m} \left( \frac{1-m}{m} \right) \cdots \left( \frac{1-(m-2)m}{m} \right) z^{\frac{1}{m}-(m-1)} \]
\[ \prod_{j=0}^{m-2} \frac{1 - jm}{z^{m-1}} \]

(this works even for \( m = 1 \)), and we arrive at

\[ \int_{\gamma} \frac{f(z)}{(z-1)^m} \, dz = \frac{2\pi i \prod_{j=0}^{m-2} (1 - jm)}{m^{m-1}(m-1)!}. \]

10. Evaluate \( \int_{\gamma} \frac{z^2 + 1}{z(z^2 + 4)} \, dz \), where \( \gamma(t) = re^{it}, 0 \leq t \leq 2\pi \), for all possible values of \( r \), \( 0 < r < 2 \), \( 2 < r < \infty \).

**Solution:** Assume first that \( 0 < r < 2 \); then \( f(z) := \frac{z^2 + 1}{z^2 + 4} \) is analytic on an open ball around 0 containing \( \gamma \), so the integral is just \( 2\pi i \cdot f(0) = \frac{\pi i}{2} \).

Now suppose \( 2 < r < \infty \). Then the function has 3 poles in any ball around 0 containing \( \gamma \), so we should use the partial fraction decomposition

\[ \frac{z^2 + 1}{z(z^2 + 4)} = \frac{1}{4z} + \frac{3z}{4(z^2 + 4)} \]

\[ = \frac{1}{4z} + \frac{3}{8} \left( \frac{1}{z + 2i} + \frac{1}{z - 2i} \right) \]

to integrate term-by-term. Each of these terms is easy: since \( \pm 2i \) are both distance 2 from 0, we can apply Proposition 2.6 to yield

\[ \int_{\gamma} \frac{1}{4z} + \frac{3}{8} \left( \frac{1}{z + 2i} + \frac{1}{z - 2i} \right) \, dz = \frac{1}{4} \int_{\gamma} \frac{1}{z} \, dz + \frac{3}{8} \left( \int_{\gamma} \frac{1}{z + 2i} \, dz + \int_{\gamma} \frac{1}{z - 2i} \, dz \right) \]

\[ = \left( \frac{1}{4} + \frac{3}{8} + \frac{3}{8} \right) 2\pi i \]

\[ = 2\pi i. \]