Math 220A HW 6 Solutions

Section 4.3

1. **Let f be an entire function and suppose there is a constant M, and $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Sow that f is a polynomial of degree $\leq n$.**

   Solution: By Liouville's theorem (which applies since f, being continuous, is bounded on the compact set $|z| \leq R$ as well), this is actually true for $n = 0$ as well. This suggests we might be able to proceed inductively. So suppose the statement is true for $n \geq 0$. f is entire, so in particular it has a derivative at 0, meaning the function

$$F(z) = \begin{cases} \frac{f(z)-f(0)}{z}, z \neq 0 \\ f'(0), z = 0 \end{cases}$$

   is well defined and continuous everywhere. Since f is entire, it has a power series expansion about 0 that converges everywhere, which gives an expansion for $F(z)$ that also converges everywhere; therefore, F is entire. Now we see that if $|f(z)| \leq M|z|^{n+1}$ for $|z| > R$, then we can take $|z| > R$, which forces

$$\begin{aligned} |F(z)| &\leq \frac{|f(z)| + |f(0)|}{|z|} \\ &\leq M|z|^n + \frac{|f(0)|}{R}. \end{aligned}$$

   Thus, if we instead take $|z| > \max(R, 1, \frac{|f(0)|}{R})$, we will have $|F(z)| \leq (M+1)|z|^n$. As a result, F is a polynomial of degree $\leq n$, so away from 0 f is a polynomial of degree at most $n + 1$. Of course, by continuity, this means f's value at 0 comes from the polynomial as well, and we are done.

   Alternate Solution: We can also use the same method of proof as that of Liouville's theorem, by showing $f^{(n+1)}(z) = 0$ for all z. To this end, if we take r sufficiently large (ie, much larger than both R and $|z|$), then for all w on the circle $\gamma(r, z)$ of radius r around z, we will have $|f(w)| \leq M|w|^n \leq M(r + |z|)^n \leq M(2r)^n$. Now we can apply Corollary 2.13 to find that

$$|f^{(n+1)}(z)| = \frac{(n+1)!}{2\pi} \left| \int_{\gamma(r,z)} \frac{f(w)}{(w-z)^{n+2}} dw \right|$$

$$\leq 2\pi r \frac{(n+1)!}{2\pi} \frac{M 2^n r^n}{r^{n+2}}$$
$$= \frac{2^n (n+1)!}{r}.$$

This is true for all large enough r, so $f^{(n+1)}(z) = 0$ for all z. It is easy to check (for example, by looking at f's power series expansion), that this forces f to be a polynomial of degree $\leq n$.

3. **Find all entire functions f such that $f(x) = e^x$ for $x \in \mathbb{R}$.**

Solution: Let f be such a function. Then $f(x) - e^x$ is an entire function that is zero on $\mathbb{R}$. Since $\mathbb{R}$ contains a limit point (it's even closed), we conclude that it is zero everywhere, and $f(z) = e^z$ is the only such function.

6. **Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic and $a \in \mathbb{G}$ such that $|f(a)| \leq |f(z)|$ for all $z \in G$. Show that either $f(a) = 0$ or a is constant.**

Solution: If $f(a) \neq 0$, then $\frac{1}{f}$ is defined and analytic on G. Observing that $|\frac{1}{f(z)}|$ is bounded above by $|\frac{1}{f(a)}|$, we can use the maximum modulus principal to say that $\frac{1}{f}$ is constant. It is also nonzero, since f is, so f is also constant.

8. **Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all z in G. Show that either $f \equiv 0$ or $g \equiv 0$.**

Solution: Suppose $f \not\equiv 0$, so there is some $a \in G$ with $f(a) \neq 0$. By continuity, we can also find an open ball $B \subset G$ around a such $f(z) \neq 0$ for $a \neq 0$. Our initial assumption then shows that g must be zero on B. B has many limit points in G, so we must have $g \equiv 0$.

9. **Let $U : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that$U(z) \geq 0$ for all $z \in \mathbb{C}$; prove that U is constant.**

Solution: Since U is harmonic on $\mathbb{C}$, we can find a harmonic conjugate $V : \mathbb{C} \to \mathbb{R}$. Let f be the resulting analytic function, and let $g(z) = e^{-f(z)}$. Then

$$|g(z)| = |e^{-u(z)}|$$
$$\leq e^0$$
$$= 1,$$

and g is constant by Liouville's theorem. Thus f, and consequently U, is constant.

10. **Show that if f and g are analytic functions on a region G such that $\bar{f}g$ is analytic then either f is constant or $g \equiv 0$.**

Solution: If $g \not\equiv 0$, we claim that it is enough to show $\bar{f}$ is analytic on a connected open subset U of G. This is true because then $\operatorname{Re} f = \frac{f+\bar{f}}{2}$ and $\operatorname{Im} f = \frac{f-\bar{f}}{2i}$ will also be analytic on U. We have already seen in an earlier homework that this can only happen if they are both constant—that is, if f is constant. But then f agrees with a constant function on an open subset of G, so as in problem 6 it must be constant on all of G.

To check that $\bar{f}$ is analytic, let $a \in G$ be such that $g(a) \neq 0$. As before, this means that we can take U to be an open ball around a contained in G, as $\bar{f} = \frac{\bar{f}g}{g}$ (which makes sense because g is nonzero on U) is the quotient of two analytic functions on U.

Section 4.4

11. **Fix $w = re^{i\theta} \neq 0$ and let γ be a rectifiable path in $\mathbb{C} - \{0\}$ from 1 to w. Show that there is an integer k such that $\int_\gamma z^{-1} dz = \log r + i\theta + 2\pi i k$.**

Solution: Choose a branch of the logarithm defined at both 1 and w. For instance, if $w \notin \mathbb{R}_{<0}$ we could take the principal branch, while if $w \in \mathbb{R}_{<0}$ we could take the branch cut at $\mathbb{R}_{\leq 0} i$ instead by setting $\log Re^{i\tau} = \log R + i\tau, -\frac{\pi}{2} < \tau < \frac{3\pi}{2}$. At any rate, we can find a region containing 1 and w such that our branch $\log z$ is defined. Let γ' be any rectifiable path from 1 to w in this region that does not go through 0. Now we have a primitive for z^{-1} around γ', so

$$\begin{aligned} \int_{\gamma'} z^{-1} dz &= \log w - \log 1 \\ &= \log r + i(\theta') + 2\pi i n \end{aligned}$$

for some n, θ' depending on what branch we took (where $re^{i\theta'} = w$). Of course, $\theta' = \theta + 2\pi m$ for some m. However, we also know that

$$2\pi i \ell := \int_{\gamma - \gamma'} z^{-1} dz$$

is just the winding number of the closed curve $\gamma - \gamma'$ around 0, and we see that

$$\begin{aligned} \int_\gamma z^{-1} dz &= \int_{\gamma'} z^{-1} dz + \int_{\gamma-\gamma'} z^{-1} dz \\ &= \log r + i\theta + 2\pi i m + 2\pi i n + 2\pi i \ell \\ &:= \log r + i\theta + 2\pi i k. \end{aligned}$$