Section 4.3

1. Let \( f \) be an entire function and suppose there is a constant \( M, \) and \( R > 0, \) and an integer \( n \geq 1 \) such that \( |f(z)| \leq M|z|^n \) for \( |z| > R. \) Show that \( f \) is a polynomial of degree \( \leq n. \)

Solution: By Liouville’s theorem (which applies since \( f, \) being continuous, is bounded on the compact set \( |z| \leq R \) as well), this is actually true for \( n = 0 \) as well. This suggests we might be able to proceed inductively. So suppose the statement is true for \( n \geq 0. \) \( f \) is entire, so in particular it has a derivative at 0, meaning the function

\[
F(z) = \begin{cases} \frac{f(z)-f(0)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}
\]

is well defined and continuous everywhere. Since \( f \) is entire, it has a power series expansion about 0 that converges everywhere, which gives an expansion for \( F(z) \) that also converges everywhere; therefore, \( F \) is entire. Now we see that if \( |f(z)| \leq M|z|^{n+1} \) for \( |z| > R, \) then we can take \( |z| > R, \) which forces

\[
|F(z)| \leq \frac{|f(z)| + |f(0)|}{|z|} \leq M|z|^n + \frac{|f(0)|}{R}.
\]

Thus, if we instead take \( |z| > \max(R, 1, \frac{|f(0)|}{R}), \) we will have \( |F(z)| \leq (M + 1)|z|^n. \) As a result, \( F \) is a polynomial of degree \( \leq n, \) so away from 0 \( f \) is a polynomial of degree at most \( n + 1. \) Of course, by continuity, this means \( f \)'s value at 0 comes from the polynomial as well, and we are done.

Alternate Solution: We can also use the same method of proof as that of Liouville’s theorem, by showing \( f^{(n+1)}(z) = 0 \) for all \( z. \) To this end, if we take \( r \) sufficiently large (ie, much larger than both \( R \) and \( |z| \)), then for all \( w \) on the circle \( \gamma(r, z) \) of radius \( r \) around \( z, \) we will have \( |f(w)| \leq M|w|^n \leq M(r + |z|)^n \leq M(2r)^n. \) Now we can apply Corollary 2.13 to find that

\[
|f^{(n+1)}(z)| = \frac{(n + 1)!}{2\pi i} \left| \int_{\gamma(r,z)} \frac{f(w)}{(w-z)^{n+2}} dw \right|
\]
\[ \leq 2\pi r \frac{(n+1)!}{2\pi} \frac{M 2^r n^n}{r^{n+2}} = 2^n (n+1)! \frac{r}{r} . \]

This is true for all large enough \( r \), so \( f^{(n+1)}(z) = 0 \) for all \( z \). It is easy to check (for example, by looking at \( f \)'s power series expansion), that this forces \( f \) to be a polynomial of degree \( \leq n \).

3. Find all entire functions \( f \) such that \( f(x) = e^x \) for \( x \in \mathbb{R} \).

**Solution:** Let \( f \) be such a function. Then \( f(x) - e^x \) is an entire function that is zero on \( \mathbb{R} \). Since \( \mathbb{R} \) contains a limit point (it's even closed), we conclude that it is zero everywhere, and \( f(z) = e^z \) is the only such function.

6. Let \( G \) be a region and suppose that \( f : G \to \mathbb{C} \) is analytic and \( a \in G \) such that \( |f(a)| \leq |f(z)| \) for all \( z \in G \). Show that either \( f(a) = 0 \) or \( a \) is constant.

**Solution:** If \( f(a) \neq 0 \), then \( \frac{1}{f} \) is defined and analytic on \( G \). Observing that \( \left| \frac{1}{f(z)} \right| \) is bounded above by \( \left| \frac{1}{f(a)} \right| \), we can use the maximum modulus principal to say that \( \frac{1}{f} \) is constant. It is also nonzero, since \( f \) is, so \( f \) is also constant.

8. Let \( G \) be a region and let \( f \) and \( g \) be analytic functions on \( G \) such that \( f(z)g(z) = 0 \) for all \( z \) in \( G \). Show that either \( f \equiv 0 \) or \( g \equiv 0 \).

**Solution:** Suppose \( f \neq 0 \), so there is some \( a \in G \) with \( f(a) \neq 0 \). By continuity, we can also find an open ball \( B \subset G \) around \( a \) such \( f(z) \neq 0 \) for \( a \neq 0 \). Our initial assumption then shows that \( g \) must be zero on \( B \). \( B \) has many limit points in \( G \), so we must have \( g \equiv 0 \).

9. Let \( U : \mathbb{C} \to \mathbb{R} \) be a harmonic function such that \( U(z) \geq 0 \) for all \( z \in \mathbb{C} \); prove that \( U \) is constant.

**Solution:** Since \( U \) is harmonic on \( \mathbb{C} \), we can find a harmonic conjugate \( V : \mathbb{C} \to \mathbb{R} \). Let \( f \) be the resulting analytic function, and let \( g(z) = e^{-f(z)} \). Then
\[
|g(z)| = |e^{-b(z)}| \leq e^0 = 1,
\]
and \( g \) is constant by Liouville's theorem. Thus \( f \), and consequently \( U \), is constant.
10. Show that if $f$ and $g$ are analytic functions on a region $G$ such that $fg$ is analytic then either $f$ is constant or $g \equiv 0$.

**Solution:** If $g \not\equiv 0$, we claim that it is enough to show $\bar{f}$ is analytic on a connected open subset $U$ of $G$. This is true because then $\text{Re} f = \frac{f + \bar{f}}{2}$ and $\text{Im} f = \frac{f - \bar{f}}{2i}$ will also be analytic on $U$. We have already seen in an earlier homework that this can only happen if they are both constant—that is, if $f$ is constant. But then $f$ agrees with a constant function on an open subset of $G$, so as in problem 6 it must be constant on all of $G$.

To check that $\bar{f}$ is analytic, let $a \in G$ be such that $g(a) \neq 0$. As before, this means that we can take $U$ to be an open ball around $a$ contained in $G$, as $\bar{f} = \frac{\bar{f}g}{g}$ (which makes sense because $g$ is nonzero on $U$) is the quotient of two analytic functions on $U$.

Section 4.4

11. Fix $w = re^{i\theta} \neq 0$ and let $\gamma$ be a rectifiable path in $\mathbb{C} - \{0\}$ from 1 to $w$. Show that there is an integer $k$ such that $\int_\gamma z^{-1}dz = \log r + i\theta + 2\pi ik$.

**Solution:** Choose a branch of the logarithm defined at both 1 and $w$. For instance, if $w \in \mathbb{R} < 0$ we could take the principal branch, while if $w \in \mathbb{R} > 0$ we could take the branch cut at $\mathbb{R} \leq 0$ instead by setting $\log \Re e^{i\tau} = \log R + i\tau, -\frac{\pi}{2} < \tau < \frac{3\pi}{2}$. At any rate, we can find a region containing 1 and $w$ such that our branch $\log z$ is defined. Let $\gamma'$ be any rectifiable path from 1 to $w$ in this region that does not go through 0. Now we have a primitive for $z^{-1}$ around $\gamma'$, so

$$\int_{\gamma'} z^{-1}dz = \log w - \log 1 = \log r + i(\theta') + 2\pi in$$

for some $n, \theta'$ depending on what branch we took (where $re^{i\theta'} = w$). Of course, $\theta' = \theta + 2\pi m$ for some $m$. However, we also know that

$$2\pi i\ell := \int_{\gamma - \gamma'} z^{-1}dz$$

is just the winding number of the closed curve $\gamma - \gamma'$ around 0, and we see that

$$\int_{\gamma} z^{-1}dz = \int_{\gamma'} z^{-1}dz + \int_{\gamma - \gamma'} z^{-1}dz$$

$$= \log r + i\theta + 2\pi im + 2\pi in + 2\pi i\ell$$

$$:= \log r + i\theta + 2\pi ik.$$