Math 220A HW 8 Solutions

Section 4.4

3. **Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z : |z| \geq R\}$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, show that $\displaystyle\int_\gamma \frac{p'(z)}{p(z)} dz = 2\pi i n$.**

   Solution: This follows quickly from the fundamental theorem of algebra: write

$$p(z) = c\prod_{i=1}^{n}(z - \alpha_i)$$

   with $c, \alpha_i \in \mathbb{C}$. By the product rule,

$$\frac{p'(z)}{p(z)} = \sum_{i=1}^{n} \frac{1}{z - \alpha_i}.$$

   so when we integrate we get $2\pi i$ times the sum of the winding numbers of all the roots with respect to γ. These are all 1 because we've chosen R so that all of the α_i are inside the circle γ (and thus in the same connected component as 0), so we end up with $2\pi i n$.

Section 4.5

5. **Let γ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \geq 2$, $\int_\gamma (z-a)^{-n} dz = 0$.**

   Solution: We don't even need to use any results from this section. As $n \geq 2$, $(z-a)^{-n}$ has a primitive (namely, $\frac{1}{1-n}(z-a)^{1-n}$) defined in a neighborhood around γ, so the integral around a closed curve must be 0.

6. **Let f be analytic on $D = B(0; 1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show $|f'(0)| \leq 1$.**

   Solution: This is Cauchy's estimate with $M = 1$.

7. **Let** $\gamma(t) = 1 + e^{it}$ **for** $0 \leq t \leq 2\pi$. **Find** $\int_\gamma \left(\frac{z}{z-1}\right)^n$ **for all positive integers** n.

Solution: Letting $f(z) = z^n$, this integral is just

$$\frac{2\pi i}{(n-1)!} f^{(n-1)}(1).$$

Of course,

$$f^{(n-1)} z^n = n!z,$$

so

$$\int_\gamma \left(\frac{z}{z-1}\right)^n = 2\pi i n.$$