Math 220A HW 9 Solutions

Section 4.6

5. **Evaluate the integral** $\displaystyle\int_\gamma \frac{dz}{z^2+1}$ **where** $\gamma(t) = 2|\cos 2\theta|e^{i\theta}$ **for** $0 \le \theta \le 2\pi$**.**

Solution: If we graph γ, we see that it makes a flower centered at the origin with petals of length 2 along the positive and negative parts of each axis. If we let $G = \mathbb{C} \setminus \{\pm i\}$, then collapsing the petals on the x-axis to the origin is a homotopy in G. Next, we can widen the two remaining petals to get a homotopy in G from γ to the union of circles of radius 1 around i and $-i$. Call these γ_+ and γ_-, respectively; then

$$\int\limits_\gamma \frac{1}{z^2+1}dz = \int\limits_{\gamma_+} \frac{1}{z^2+1}dz + \int\limits_{\gamma_-} \frac{1}{z^2+1}dz.$$

Now we can use the partial fraction decomposition

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

and the fact that $\frac{1}{z\pm 1}$ is analytic on $\gamma_\pm$ (so that their integrals over the respective curves are zero) to get that

$$\begin{aligned}\int\limits_\gamma \frac{1}{z^2+1}dz &= \frac{1}{2i}\Big(\int\limits_{\gamma_+} \frac{1}{z-i}dz - \int\limits_{\gamma_-} \frac{1}{z+i}dz\Big)\\ &= \pi(n(\gamma_+; i) - n(\gamma_-; -i))\\ &= \pi(1-1)\\ &= 0.\end{aligned}$$

6. **Let $\gamma(\theta) = \theta e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ and $\gamma(\theta) = 4\pi - \theta$ for $2\pi \leq \theta \leq 4\pi$. Evaluate $\displaystyle\int_\gamma \frac{dz}{z^2+\pi^2}$.**

Solution: Let $f = \frac{1}{z^2+\pi^2}$. γ is a (counterclockwise) spiral from 0 to 2π, then a line segment from 2π to 0. Notice that the resulting closed curve only encloses one of the poles of f—namely, $-i\pi$—and in fact is homotopic in $G := \mathbb{C} \setminus \{\pm\pi i\}$ to a circle of radius π around that pole. If we let τ be this circle, then $g(z) := \frac{1}{z-i\pi}$ is analytic in a disc containing τ, so we can use Cauchy's formula to find that

$$\begin{aligned}\int_\tau \frac{1}{z^2+\pi^2}dz &= \int_\tau \frac{g(z)}{z+i\pi}dz \\ &= 2\pi i \cdot g(-i\pi) \\ &= -1.\end{aligned}$$

Now $\gamma \sim \tau$ in G implies [1] that

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+\pi^2} &= \int_\tau \frac{dz}{z^2+\pi^2} \\ &= -1\end{aligned}$$

10. **Find all possible values of $\displaystyle\int_\gamma \frac{dz}{1+z^2}$ where γ is any closed rectifiable curve in $\mathbb{C}$ not passing through $\pm i$.**

Solution: As before, we can take partial fractions to write

$$\begin{aligned}\int_\gamma \frac{dz}{1+z^2} &= \frac{1}{2i}\int_\gamma (\frac{1}{z-i} - \frac{1}{z+i}) \\ &= \pi(n(\gamma; i) - n(\gamma; -i)).\end{aligned}$$

Thus (taking γ to be 0 or a circle wrapping n times around $\pm i$, for instance), $\int_\gamma \frac{dz}{1+z^2}$ can be π times any integer. If you've taken or are taking topology, this is related to the fact that the fundamental group of $\mathbb{C} \setminus \{\pm i\}$ is $\mathbb{Z} * \mathbb{Z}$.

[1] as f is analytic in G by construction

Section 4.7

2. **Let G be open and suppose that γ is a closed rectifiable curve in G such that $\gamma \approx 0$. Set $r = d(\{\gamma\}, \partial G)$, and $H = \{z \in \mathbb{C} : n(\gamma; z) = 0\}$.**

   (a) **Show that $\{z : d(z, \partial G) < \frac{1}{2}r\} \subset H$.**

   Solution: Let S be this set. If $S = \emptyset$ (ie, if $\partial G = \emptyset$) this is obvious; otherwise, suppose $z \in S$. In $\mathbb{C} \setminus \{\gamma\}$, z must lie in the same connected component as some $w \in \partial G$ because z is closer to ∂G than γ. Therefore, they have the same winding number with respect to γ. Since G is open, w is not in G, so by assumption

   $$n(\gamma; z) = n(\gamma; w) = 0,$$

   and $z \in H$. Thus $S \subset H$.

   (b) **Use part (a) to show that if $f : G \to \mathbb{C}$ is analytic then $f(z) = \alpha$ has at most a finite number of solutions z such that $n(\gamma; z) \neq 0$.**

   Solution: This is false unless we assume f is nonconstant. By replacing f with $f - \alpha$ (which is also analytic), we can assume $\alpha = 0$. We already know that $n(z; \gamma)$ can only be nonzero on the bounded compenents of $\mathbb{C} \setminus \{\gamma\}$, so in particular the set Z of zeroes of f with nonzero winding number is bounded. However, we also know Z is discrete since f is nonconstant.

   We claim Z is also closed. To see this, suppose that z is a limit point of Z. If $z \notin Z$, then it cannot be in G either (again because f is nonconstant), so it must be in ∂G. But then we could find $z_0 \in Z$ within $\frac{1}{2}r$ of z, which by part (a) is impossible. Thus Z is closed since it contains all its limit points.

   Finally, Z is compact since it is closed and bounded, and any discrete compact set is finite.

3. **Let f be analytic in $B(a; R)$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity m iff $f^{(m-1)}(a) = \cdots = f(a) = 0$ and $f^{(m)}(a) \neq 0$.**

   Solution: Both directions follow from the power series expansion of f around a.