1. Classification of singularities

Defn 1.1 A function f has an isolated singularity at $z=a$ if there is an $R>0$ s.t f is defined and analytic in $B(a;R)-\{a\}$ but not in $B(a;R)$.

The pt a is called a removable singularity if there is an analytic function $g: B(a;R) \to \mathbb{C}$ s.t $g(z) = f(z)$ for $0<|z-a|<R$.

E.g.: $f_1 = \frac{\sin z}{z}$, $f_2 = \frac{1}{z}$, $f_3 = e^{\frac{1}{z}}$ all have isolated singularities at $z=0$. However, only f_1 has a removable singularity.

Thm. If f has an isolated singularity at a then the pt $z=a$ is a removable singularity iff

$$\lim_{z \to a} (z-a) f(z) = 0$$

Pf: Suppose f is analytic in $\{z: 0<|z-a|<R\}$.

Define
$$g(z) = \begin{cases} (z-a) f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a. \end{cases}$$

Then g is continuous.

If we can prove g is analytic, then a is a removable singularity of f. Indeed, if g is analytic, noting a is a zero of g, we have $g(z) = (z-a) h(z)$ for some analytic function h defined on $B(a;R)$. But then $h(z) = f(z)$ on $0<|z-a|<R$.

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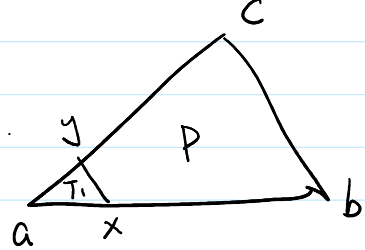
To show g is analytic on $B(a;R)$, we will apply Morera's Thm. Let T be a triangle in $B(a;R)$ and let Δ be the inside of T union T

(1) If $a \notin \Delta$ then $T \sim 0$ in $\{z: 0<|z-a|<R\}$.

Consequently, $\int_T g = 0$.

(2) If a is an vertex of T, say $T = [a, b, c, a]$.



we have

$$\int_T g = \int_{T_1} g + \int_P g$$

By Cauchy's thm, $\int_P g = 0$

Also, in T_1, as $\lim_{z\to a} g = 0$. $\Rightarrow$

$$\left| \int_{T_1} g \right| \le \int_{T_1} |g| \, |dz| \to 0 \text{ when } T_1 \to a$$

Hence $\int_T g = 0$

Defn 1.3. If $z=a$ is an isolated singularity of f, then a is a pole of f if $\lim_{z\to a} |f(z)| = \infty$.

An isolated singularity is called an essential singularity if it is neither a pole nor a removable singularity.

E.g. ① $(z-a)^{-m}$ has a pole at $z=a$ for $m \ge 1$

② Consider $e^{\frac{1}{z}}$. $z=0$ is an essential singularity.

Remark f has a pole at $z=a \Rightarrow f^{-1}$ has a

Remark: f has a pole at $z=a \Rightarrow f^{-1}$ has a removable singularity at $z=a$.

Proposition 1.4: If G is a region with a in G and if f is analytic on $G-\{a\}$ with a pole at $z=a$ then there is a unique positive integer m and an analytic function $g: G \to \mathbb{C}$ s.t

$$f(z) = \frac{g(z)}{(z-a)^m}; \quad g(a) \neq 0 \quad (*)$$

Defn 1.6: If f has a pole at $z=a$ and m is the smallest positive integer s.t $f(z)(z-a)^m$ has a removable singularity at $z=a$ then f has a pole of order m at $z=a$.

Remark: (1) The order m is the same integer as in $(*)$.

(2) Let f have a pole of order m at $z=a$, put $f(z) = g(z)(z-a)^{-m}$. Since g is analytic in a disk $B(a;R)$ it has a power series expansion at a. Let

$$g(z) = A_m + A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k (z-a)^k.$$

Hence

$$f(z) = \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)} + g_1(z) \quad (**)$$

where g_1 is analytic in $B(a;R)$ and $A_m \neq 0$.

Defn 1.8: If f has a pole of order m at $z=a$ and f satisfies $(**)$ then $A_m(z-a)^{-m} + \cdots + A_1(z-a)^{-1}$

f satisfies (**) then $A_m(z-a)^{-m}+\cdots+A_1(z-a)^{-1}$ is called the singular part of f at $z=a$.

Defn 1.10. (1) $\sum_{n=-\infty}^{\infty} z_n$ is absolutely convergent if both $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ are absolutely convergent.

In this case, $\sum_{n=-\infty}^{\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n$.

(2) If u_n is a function on a set S for $n\in\mathbb{Z}$. then $\sum_{n=-\infty}^{\infty} u_n$ converges uniformly if both $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=1}^{\infty} u_{-n}$ converges uniformly.

Defn: Let $0\le R_1<R_2\le\infty$ and a is any complex number. define $ann(a;R_1,R_2)=\{z: R_1<|z-a|<R_2\}$.

Note $ann(a;0,R_2)$ is a punctured disk.

Thm 1.11 (Laurent Series expansion)

Let f be analytic in the annulus $ann(a;R_1,R_2)$.

Then
$$f(z)=\sum_{n=-\infty}^{+\infty} a_n(z-a)^n.$$

where the convergence is absolute and uniform over $\overline{ann(a;r_1,r_2)}$ if $R_1<r_1<r_2<R_2$. Also the coefficients a_n are given by the formula
$$a_n=\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$
where γ is the circle $|z-a|=r$ for any r, $R_1<r<R_2$.

Moreover, the series is unique.

Pf: Page 107 in textbook.

Corollary 1.18: Let $z=a$ be an isolated singularity of f and let $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ be its Laurent Expansion in $\text{ann}(a;0,R)$. Then:

(a) $z=a$ is a removable singularity iff $a_n = 0$ for $n \le -1$;

(b) $z=a$ is a pole of order m iff $a_{-m} \neq 0$ and $a_n = 0$ for $n \le -(m+1)$;

(c) $z=a$ is an essential singularity iff $a_n \neq 0$ for infinitely many negative integers n.

Thm 1.21: (Casorati - Weierstrass Thm) If f has an essential singularity at $z=a$ then for every $\delta > 0$, $\{f[\text{ann}(a;0,\delta)]\}^- = \mathbb{C}$.

Pf: Suppose that f is analytic in $\text{ann}(a;0,R)$. We will show that for any given $c \in \mathbb{C}$ and $\varepsilon > 0$, we can find a z with $|z-a| < \delta$ and $|f(z) - c| < \varepsilon$. Suppose Not. That is, assume there is a $c \in \mathbb{C}$ and $\varepsilon > 0$ s.t $|f(z) - c| \ge \varepsilon$ for all z in $G \triangleq \text{ann}(a;0,\delta)$. Thus $\lim_{z \to a} |z-a|^{-1} |f(z) - c| = \infty$, which implies

$$(z-a)^{-1}(f(z) - c) \text{ has a pole at } z = a.$$

If m is the order of this pole then

$$\lim_{z \to a} |z-a|^m \, |f(z) - c| = 0$$

$$\Rightarrow \quad |z-a|^{m+1} |f(z)| \le |z-a|^{m+1} |f(z) - c| + |z-a|^{m+1} |c|$$

Thus $\lim_{z \to a} |z-a|^{m+1} |f(z)| = 0$ since $m \ge 1$.

But this implies $f(z)(z-a)^m$ has a removable singularity and thus $f(z)$ has a pole at $z=a$.

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This contradicts the hypothesis.