2. Residues

Defn 2.1: Let f have an isolated singularity at $z=a$ and let

$$f(z)=\sum_{n=-\infty}^{+\infty} a_n(z-a)^n \qquad (1)$$

be its Laurent Expansion about $z=a$. Then the residue of f at $z=a$ is the coefficient a_{-1}. We denote it by

$$\mathrm{Res}(f;a)=a_{-1}.$$

Recall: If f is analytic in $\mathrm{ann}(a;R_1,R_2)$, then the coefficient a_{-1} in (1) is given by

$$a_{-1}=\frac{1}{2\pi i}\int_\gamma f(z)\,dz.$$

where γ is the circle $|z-a|=r$ for any $R_1<r<R_2$.

Thm 2.2 (Residue Thm)

Let f be analytic in the region G except for the isolated singularities $a_1,\cdots,a_m$. If γ is a closed rectifible curve in G which does not pass through any of the pts a_k and if $\gamma\approx 0$ in G then

$$\frac{1}{2\pi i}\int_\gamma f=\sum_{k=1}^{m} n(\gamma;a_k)\,\mathrm{Res}(f;a)$$

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Pf: Let $m_k = n(\gamma;a_k)$ for $1\le k\le m$, then choose $r_1,\cdots,r_m$ s.t the disks $B(a_k;r_k)$ are disjoint, none of them intersects $\{\gamma\}$, and each disk is contained in G.

Let $\gamma_k(t) = a_k + r_k\exp(-2\pi i m_k t)$ for $0\le t\le 1$.

Then for $1\le j\le m$,

$$n(\gamma;a_j) + \sum_{k=1}^{m} n(\gamma_k;a_j) = 0.$$

Since $\gamma \approx 0\,(G)$ and $\overline{B}(a_k;r_k)\subset G$

$$n(\gamma;a) + \sum_{k=1}^{m} n(\gamma_k;a) = 0$$

for $a\notin G$ or $a\in\{a_1,\cdots,a_m\}$, i.e. $a\notin G-\{a_1,\cdots,a_m\}$.

Since f is analytic in $G-\{a_1,\cdots,a_m\}$.

Thm IV.5.7 gives

$$0 = \int_\gamma f + \sum_{k=1}^{m}\int_{\gamma_k} f$$

If $f(z) = \sum_{-\infty}^{+\infty} b_n (z-a_k)^n$ is the Laurent expansion about $z=a_k$ then this series converges uniformly on $\partial B(a_k,r_k)$.

Hence

$$\int_{\gamma_k} f = \sum_{-\infty}^{\infty} b_n \int_{\gamma_k}(z-a_k)^n \qquad (2)$$

But $\int_{\gamma_k}(z-a_k)^n = 0$ if $n\neq -1$ as $(z-a_k)^n$ has a

$\int (z-a_k)^{-1} = 2\pi i\, n(\gamma_k;a_k)\,\mathrm{Res}(f;a_k)$.

$\int_{\gamma_k}$ ... primitive. Also $b_{-1} \int_{\gamma_k} (z-a_k)^{-1} = 2\pi i\, n(\gamma_k; a_k) \operatorname{Res}(f; a_k)$.

Hence (2) implies the desired result.

Remark: The Residue Thm is often used to compute certain line integrals.

Suppose f has a pole of order $m \geq 1$ at $z=a$. Then we have

$$f(z) = \frac{b_0}{(z-a)^m} + \cdots + \frac{b_{m-1}}{(z-a)} + \sum_{k=0}^{\infty} b_{m+k} (z-a)^k .$$

Note then $\operatorname{Res}(f; a) = b_{m-1}$.

proposition 2.4: Suppose f has a pole of order m at $z=a$ and put $g(z) = (z-a)^m f(z)$, then

$$\operatorname{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a) .$$

In particular, if $z=a$ is a simple pole, then

$$\operatorname{Res}(f; a) = g(a) = \lim_{z \to a} (z-a) f(z).$$

Example 2.5. Show $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4}\, dx = \frac{\pi}{\sqrt{2}}$

Let $f(z) = \frac{z^2}{1+z^4}$. Then f has its poles the fourth root of -1. There are root of -1. There are

$$a_1 = e^{i\frac{\pi}{4}}, \quad a_2 = e^{i\frac{3\pi}{4}}, \quad a_3 = e^{i\frac{5\pi}{4}}, \quad a_4 = e^{i\frac{7\pi}{4}}$$

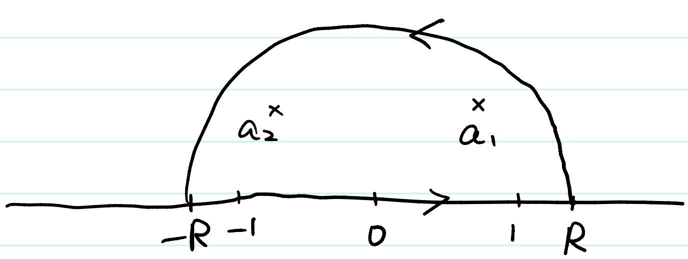
Each a_n is a simple pole of f. Moreover,

$$\begin{aligned}
\operatorname{Res}(f; a_1) &= \lim_{z \to a_1} (z - a_1) f(z) \\
&= a_1^2 (a_1 - a_2)^{-1} (a_1 - a_3)^{-1} (a_1 - a_4)^{-1} \\
&= \frac{1 - i}{4\sqrt{2}} = \frac{1}{4} e^{-\frac{\pi i}{4}}
\end{aligned}$$

Similarly

$$\operatorname{Res}(f; a_2) = \frac{-1 - i}{4\sqrt{2}} = \frac{1}{4} e^{-\frac{3\pi}{4} i}$$

Now let $R > 1$ and let γ be the closed path which is the boundary of the upper half of the disk of radius R with center zero, traversed in the counter-clockwise direction.



$$\begin{aligned}
\frac{1}{2\pi i} \int_\gamma f &= \operatorname{Res}(f; a_1) + \operatorname{Res}(f; a_2) \\
&= \frac{-i}{2\sqrt{2}}.
\end{aligned}$$

But, applying the defn of line integral

$$\frac{1}{2\pi i} \int_\gamma f = \frac{1}{2\pi i} \int_{-R}^{R} \frac{x^2}{\cdots} dx + \frac{1}{2\pi i} \int_0^{\pi} \frac{R^3 e^{3it}}{\cdots} dt$$

$$\frac{1}{2\pi i}\int_\gamma f = \frac{1}{2\pi i}\int_{-R}^{R}\frac{x^2}{1+x^4}\,dx + \frac{1}{2\pi}\int_0^\pi \frac{R^3e^{3it}}{1+R^4e^{4it}}\,dt$$

This gives
$$\int_{-R}^{R}\frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}} - iR^3\int_0^\pi \frac{e^{3it}}{1+R^4e^{4it}}\,dt$$

Note $|1+R^4e^{4it}| \geq R^4-1$. Therefore,
$$\left|iR^3\int_0^\pi \frac{e^{3it}}{1+R^4e^{4it}}\,dt\right| \leq \frac{\pi R^3}{R^4-1} \to 0 \text{ when } R\to\infty.$$

Thus
$$\int_{-\infty}^{+\infty}\frac{x^2}{1+x^4}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{1+x^4}\,dx$$
$$= \frac{\pi}{\sqrt{2}}.$$

Example 2.7. Show that for $a>1$
$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}$$

If $z=e^{i\theta}$ then $\bar{z}=\frac{1}{z}$ and so
$$a+\cos\theta = a+\frac{1}{2}(z+\bar{z}) = a+\frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+2az+1}{2z}$$

Hence
$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{a+\cos\theta}$$
$$= -i\int_\gamma \frac{dz}{z^2+2az+1}$$

$\left(\text{Here } \int_\pi^{2\pi}\frac{d\theta}{a+\cos\theta} = \int_0^\pi \frac{du}{a+\cos u} \qquad u=2\pi-\theta\right)$

where γ is the circle $|z|=1$. But

$$z^2+2az+1=(z-\alpha)(z-\beta) \quad \text{with}$$

$$\begin{cases} \alpha = -a+(a^2-1)^{\frac{1}{2}} \\ \beta = -a-(a^2-1)^{\frac{1}{2}} \end{cases}$$

Since $a>1$ it follows that $|\alpha|<1$ and $|\beta|>1$.

Note $\quad \operatorname{Res}(f,\alpha) = \lim_{z\to\alpha} (z-\alpha) f = \frac{1}{\alpha-\beta} = \frac{1}{2(a^2-1)^{\frac{1}{2}}}$.

By the Residue Thm

$$\int_\gamma \frac{dz}{z^2+2az+1} = \frac{\pi i}{\sqrt{a^2-1}}$$