2. Residues

Defn 2.1: Let \( f \) have an isolated singularity at \( z = a \) and let
\[
 f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad (1)
\]
be its Laurent Expansion about \( z = a \). Then the residue of \( f \) at \( z = a \) is the coefficient \( a_{-1} \). We denote it by
\[
 \text{Res}(f; a) = a_{-1}. 
\]

Recall: If \( f \) is analytic in \( \text{ann}(a; R_1, R_2) \), then the coefficient \( a_{-1} \) in (1) is given by
\[
 a_{-1} = \frac{1}{2\pi i} \oint_{C} f(z) \, dz
\]
where \( C \) is the circle \( |z-a|=r \) for any \( R_1 < r < R_2 \).

Thm 2.2 (Residue Thm)

Let \( f \) be analytic in the region \( G \) except for the isolated singularities \( a_1, \ldots, a_m \). If \( \gamma \) is a closed rectifiable curve in \( G \) which does not pass through any of the pts \( a_k \) and if \( \gamma \not\subset \{a\} \) in \( G \) then
\[
 \frac{1}{2\pi i} \oint_{\gamma} f = \sum_{k=1}^{m} n(\gamma; a_k) \text{Res}(f; a_k)
\]
\[ \frac{1}{2\pi i} \int_{\partial} f = \sum_{k=1}^{m} n(\sigma; a_k) \text{Res}(f; a) \]

**Proof:** Let \( m_k = n(\sigma; a_k) \) for \( 1 \leq k \leq m \), then choose \( r_1, \ldots, r_m \) so that the disks \( B(a_k; r_k) \) are disjoint, none of them intersect \( \partial \), and each disk is contained in \( G \).

Let \( \gamma(t) = a_k + r_k \exp(-2\pi i m_k t) \) for \( 0 \leq t \leq 1 \).

Then for \( 1 \leq j \leq m \),
\[ n(\sigma; a_j) + \sum_{k=1}^{m} n(\sigma_k; a_j) = 0. \]

Since \( \sigma \notin \partial G \) and \( \partial B(a_k; r_k) \subset G \),
\[ n(\sigma; a) + \sum_{k=1}^{m} n(\sigma_k; a) = 0 \]
for \( a \notin G \) or \( a \in \{a_1, \ldots, a_m\} \), i.e. \( a \notin G - \{a_1, \ldots, a_m\} \).

Since \( f \) is analytic in \( G - \{a_1, \ldots, a_m\} \).

Thm IV.5.7 gives
\[ 0 = \int_{\partial} f + \sum_{k=1}^{m} \int_{\partial k} f. \]

If \( f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a_k)^n \) is the Laurent expansion about \( z = a_k \), then this series converges uniformly on \( \partial B(a_k; r_k) \).

Hence
\[ \int_{\partial} f = \sum_{n=-\infty}^{\infty} b_n \int_{\partial_k} (z-a_k)^n. \]

But \( \int_{\partial_k} (z-a_k)^n = 0 \) if \( n \neq -1 \) as \( (z-a_k)^n \) has a

Hence \( n = -1 \) \( \int_{\partial_k} (z-a_k)^{-1} = 2\pi i n(\sigma_k; a_k) \) \( \text{Res}(f; a_k) \).
\[ \text{Remark: The Residue Thm is often used to compute certain line integrals.} \]

Suppose \( f \) has a pole of order \( m \geq 1 \) at \( z = a \). Then we have

\[ f(z) = \frac{b_0}{(z-a)^m} + \cdots + \frac{b_{m-1}}{(z-a)^1} + \sum_{k=0}^{\infty} b_{m+k}(z-a)^k. \]

Note then \( \text{Res}(f; a) = b_{m-1} \).

**Proposition 2.4:** Suppose \( f \) has a pole of order \( m \) at \( z = a \) and put \( g(z) = (z-a)^m f(z) \), then

\[ \text{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a). \]

In particular, if \( z = a \) is a simple pole, then

\[ \text{Res}(f; a) = g(a) = \lim_{z \to a} (z-a)f(z). \]

**Example 2.5.** Show \( \int_{-\infty}^{+\infty} \frac{x^2}{1 + x^4} \, dx = \frac{\pi}{\sqrt{2}} \)

Let \( f(z) = \frac{z^2}{1 + z^4} \). Then \( f \) has its poles the fourth root of \(-1\). There are
root of $-1$. There are

$$a_1 = e^{i\frac{\pi}{4}}, \quad a_2 = e^{i\frac{3\pi}{4}}, \quad a_3 = e^{i\frac{5\pi}{4}}, \quad a_4 = e^{i\frac{7\pi}{4}}$$

Each $a_n$ is a simple pole of $f$. Moreover,

$$\text{Res}(f; a_1) = \lim_{z \to a_1} (z - a_1)f(z)$$

$$= a_1^2 (a_1 - a_2)^{-1} (a_1 - a_3)^{-1} (a_1 - a_4)^{-1}$$

$$= \frac{1-i}{4\sqrt{2}} = \frac{1}{4} e^{-\frac{i\pi}{4}}$$

Similarly,

$$\text{Res}(f; a_2) = \frac{-1-i}{4\sqrt{2}} = \frac{1}{4} e^{-\frac{3\pi}{4}}$$

Now let $R > 1$ and let $\gamma$ be the closed path which is the boundary of the upper half of the disk of radius $R$ with center zero, traversed in the counter-clockwise direction.

\[
\frac{1}{2\pi i} \oint_{\gamma} f = \text{Res}(f; a_1) + \text{Res}(f; a_2)
\]

\[
= \frac{-1}{2\sqrt{2}}.
\]

But, applying the defn of line integral

\[
\int \int_0^1 \int_R x^2 \, d\lambda + \frac{R^3}{1!} e^{3it} \, dt
\]
\[
\frac{1}{2\pi i} \oint_C f = \frac{1}{2\pi i} \int_{-R}^{R} \frac{x^2}{1+x^4} \, dx + \frac{1}{2\pi i} \int_0^{\pi} \frac{R^3 e^{3it}}{1+R^4 e^{4it}} \, dt
\]

This gives
\[
\int_{-R}^{R} \frac{x^2}{1+x^4} \, dx = \frac{\pi}{\sqrt{2}} - i R^3 \int_0^{\pi} \frac{e^{3it}}{1+R^4 e^{4it}} \, dt
\]

Note \( |1+R^4 e^{4it}| \geq R^4 - 1 \). Therefore,
\[
\left| i R^3 \int_0^{\pi} \frac{e^{3it}}{1+R^4 e^{4it}} \, dt \right| \leq \frac{\pi R^3}{R^4 - 1} \to 0 \text{ when } R \to \infty
\]

Thus
\[
\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{x^2}{1+x^4} \, dx
\]
\[
= \frac{\pi}{\sqrt{2}}.
\]

Example 2.7. Show that for \( a > 1 \)
\[
\int_0^{\pi} \frac{d\theta}{a + \cos \theta} = \frac{\pi}{\sqrt{a^2 - 1}}
\]

If \( z = e^{i\theta} \) then \( \overline{z} = \frac{1}{z} \) and so
\[
a + \cos \theta = a + \frac{1}{2} (z + \overline{z}) = a + \frac{1}{2} (z + \frac{1}{z}) = \frac{z^2 + 2az + 1}{2z}
\]

Hence
\[
\int_0^{\pi} \frac{d\theta}{a + \cos \theta} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{a + \cos \theta}
\]
\[
= -i \int_{\gamma} \frac{dz}{z^2 + 2az + 1}
\]

(Here \( \int_{\gamma}^{2\pi} \frac{du}{u + a} = \int_0^\pi \frac{du}{a + \cos u} \quad u = 2\pi - \theta \))
Where \( \gamma \) is the circle \( |z| = 1 \). But

\[
z^2 + 2az + 1 = (z - \alpha)(z - \beta) \quad \text{with}
\]
\[
\begin{align*}
\alpha &= -a + (a^2 - 1)^{1/2} \\
\beta &= -a - (a^2 - 1)^{1/2}
\end{align*}
\]

Since \( a > 1 \) it follows that \( |\alpha| < 1 \) and \( |\beta| > 1 \).

Note

\[
\operatorname{Res}(f, \alpha) = \lim_{z \to \alpha} (z - \alpha) f = \frac{1}{2} = \frac{1}{2(a^2 - 1)^{1/2}}.
\]

By the Residue Thm

\[
\int_{\gamma} \frac{dz}{z^2 + 2az + 1} = \frac{\pi i}{\sqrt{a^2 - 1}}
\]