§1. The maximum Principle

Recall open mapping Theorem:

Assume f non-constant analytic on a region G.
Then for any open subset U in G. $f(U)$ is open
In particular, $\forall z \in G$, $f(z)$ is an interior point of $f(G)$

Thm 1.1. Maximum Modulus Thm (1st version)

If f is analytic in a region G and a is a point in G s.t $|f(a)| \geq |f(z)|$ for all $z \in G$, then f must be a constant function on G.

Pf. We prove its contrapositive.

f is not constant $\Rightarrow$ $|f(a)|$ cannot be the maximum modulus

This is clear from open mapping thm. Indeed, by open mapping thm, $f(a)$ is an interior pt of $f(G)$.

Thm 1.2. Maximum Modulus Thm (2nd Version)

Let G be a bounded open set in $\mathbb{C}$. Suppose f is a continuous function on $\overline{G}$ which is analytic on G, is a continuous function on $\overline{G}$ which is analytic on G, then $\max\{|f(z)| : z \in \overline{G}\} = \max\{|f(z)| : z \in \partial G\}$ (1)

Pf: Since $\overline{G}$ is compact and f is continuous on $\overline{G}$,
$\Rightarrow \exists\ a \in \overline{G}$. s.t $|f(a)| \geq |f(z)|$, $\forall z \in \overline{G}$.

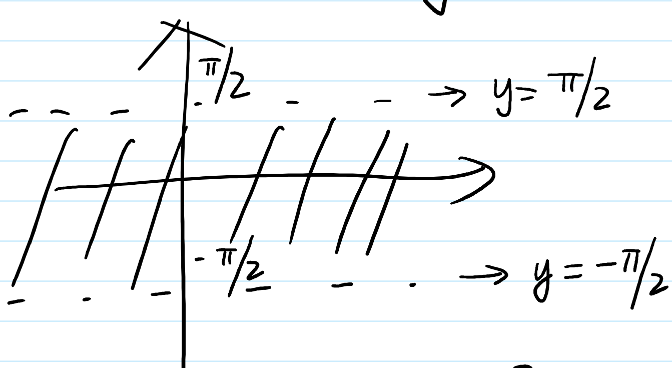
Case I: If $a \in G$, by 1st version of M.M.T., f is constant. $\Rightarrow$ (1) holds.

Case II. If $a \notin G$, then $a \in \partial G$. $\Rightarrow$ (1) holds.

Q: Does the conclusion still holds if G is unbounded?

A: No! See the following example.

E.g1: Let $G = \{z = x+iy : -\frac{1}{2}\pi < y < \frac{1}{2}\pi\}$.



put $f(z) = e^{e^{z}}$, which is analytic in $\mathbb{C}$.

Note $|f(z)| = |e^{e^z}| = |e^{\mathrm{Re}\, e^z}| = |e^{e^x \cos y}|$

$\Rightarrow$ on ∂G, $y = \pm \pi/2$, $\cos y = 0 \Rightarrow |f(z)| = |e^0| = 1$

$\Rightarrow \quad \max\{|f(z)| : z \in \partial G\} = 1$

But when $z = x \in \mathbb{R}$,

$$|f(z)| = |e^{e^x}| \to \infty \text{ as } x \to +\infty .$$

Thus (1) fails.

Remark: However, we have the following M.M.T for an bounded domain.

(A simpler version for) The Phragmèn-Lindelöf Thm:

Let $G = \{z = x+iy : \alpha < y < \beta\}$. Let f be continuous on $\overline{G}$ and analytic in G. Assume

(i) $|f| \le M$ on $\partial G = \{y = \alpha\} \cup \{y = \beta\}$.

(ii) $|f|$ is bounded on $\overline{G}$

Then $|f| \le M$ on G.

<u>Defn</u> 1.3: Let $g : G \to \mathbb{R}$ and $a \in \overline{G}$ or $a = \infty$.

Define

If $a \in \overline{G}$, $\limsup\limits_{z \to a} g(z) = \lim\limits_{r \to 0^+} \sup\{g(z) : z \in G \cap B(a;r)\}$

If $a=\infty$, $\limsup_{z\to a} g(z) = \limsup_{z\to\infty} g(z)$

$$= \lim_{R\to+\infty} \sup\{g(z) : z\in G\cap\{|z|>R\}\}.$$

Define $\liminf_{z\to a} g(z)$ similarly (Replace all "sup" by "inf" in the above)

Def^n: Let $G\subset\mathbb{C}$. Define the extended boundary $\partial_\infty G$ of G to be the boundary of G in $\mathbb{C}_\infty=\mathbb{C}\cup\{\infty\}$. In other words,

$$\partial_\infty G = \partial G \text{ if } G \text{ is bounded},$$
$$\partial_\infty G = \partial G\cup\{\infty\} \text{ if } G \text{ is unbounded}.$$

Thm 1.4 Maximum Modulus Thm (3rd version)

Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose there is a constant $M>0$ s.t $\limsup_{z\to a}|f(z)|\le M$ for all $a\in\partial_\infty G$. Then $|f(z)|\le M$ for all $z\in G$.

Pf: P_{129} in the book.

Remark: In this 3rd version, "$\partial_\infty G$" cannot be replaced by "∂G":

Let G and f be in E.g 1. Then $\limsup_{z\to a}|f(z)|=1$

Let G and f be in E.g 1. Then $\limsup_{z\to a}|f(z)| = 1$ for all $a \in \partial G$. But $|f(z)|$ is unbounded on G.

Note $\limsup_{z\to\infty} |f(z)| = +\infty$