§2 Schwarz's Lemma

Schwarz's lemma: Let $\mathbb{D} = \{z : |z| < 1\}$ and suppose f is analytic on D with

(a) $|f(z)| \le 1$ for $z \in D$

(b) $f(0) = 0$.

Then $|f'(0)| \le 1$ and $|f(z)| \le |z|$ for all $z \in \mathbb{D}$.

Moreover if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some $z \neq 0$, then $\exists\ c \in \mathbb{C}$ with $|c| = 1$ s.t $f(z) = cz$, $\forall z \in D$.

Pf: Define $g : \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Note $\lim_{z \to 0} \frac{f(z)}{z} = f'(0) \Rightarrow g$ is analytic on $\mathbb{D}$.

Claim: $\forall z_0 \in D$, $|g(z_0)| \le 1$

Pf of claim: Let $|z_0| < r < 1$. Then $z_0 \in B(0;r)$. Then by M.M.T

$$|f(z_0)| \le \max\{|g(z)| : z \in \partial B(0;r)\}$$

But for $z \in \partial B(0;r)$, $|g(z)| = \left|\frac{f(z)}{z}\right| \le \frac{1}{r}$

$\Rightarrow |f(z_0)| \le \frac{1}{r}$

Let $r \to 1 \Rightarrow |f(z_0)| \le 1$

By claim, If $z_0 \neq 0 \Rightarrow |f(z_0)| \le |z_0|$

If $z_0 = 0 \Rightarrow |f'(0)| \le 1$

If $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \neq 0$,

then g attains its maximum in $\mathbb{D}$

By M.M.T, g is constant. and thus $g \equiv c$ for some $|c| = 1$.

Recall Let $G \subset \mathbb{C}$ open

Defn: If $f: G \to G$: analytic, one-to-one, onto,

then we call f an automorphism of G.

$\mathrm{Aut}(G)$ = the set of automorphisms of G.

Write for every $a \in \mathbb{D}$,

$$\varphi_a(z) \triangleq \frac{z-a}{1-\bar{a}z} \Rightarrow \varphi_a \text{ is analytic on } \left\{ |z| < \frac{1}{|a|} \right\}.$$

proposition (1) $\varphi_a(\varphi_{-a}(z)) = \varphi_{-a}(\varphi_a(z)) = z$ for $|z| \le 1$.

(2) $\varphi_a(\partial\mathbb{D}) = \partial\mathbb{D}$

(3) $\varphi_a \in \mathrm{Aut}(\mathbb{D})$

(4) $\varphi_a(a) = 0$, $\varphi_a(0) = -a$

(5) $\varphi_a'(0) = 1 - |a|^2$, $\varphi_a'(a) = (1-|a|^2)^{-1}$.

Pf: (1) Exercise

(2) Let $z = e^{i\theta} \in \partial\mathbb{D} \Rightarrow$

$$|\varphi_a(e^{i\theta})| = \left|\frac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}}\right| = \left|\frac{e^{i\theta}-a}{e^{-i\theta}-\bar{a}}\right| = 1$$

$\Rightarrow \varphi_a(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$

This equality follows from (1).

(3) Since $\varphi_a(\partial\mathbb{D}) \subseteq \partial\mathbb{D} \overset{M.M.T}{\Longrightarrow} |\varphi_a(z)| < 1$ for $z \in \mathbb{D}$.

$\Rightarrow \varphi_a : \mathbb{D} \to \mathbb{D}$

By (1), it is one-to-one, and onto.

(4), (5) Exercise.

We will see

$$\mathrm{Aut}(\mathbb{D}) = \{c\varphi_a : |c| = 1,\ a \in \mathbb{D}\}$$

Thm: If f is an automorphism of $\mathbb{D}$ with $f(a) = 0$, then $\exists\ c \in \mathbb{C}$ with $|c| = 1$ such that

$$f = c\varphi_a$$

Remark: The above representation is unique. That is,

$$c\varphi_a = \hat{c}\varphi_{\hat{a}}$$
$$\Longleftrightarrow a = \hat{a},\ c = \hat{c}.$$

Pf of Thm: Let $g \triangleq f \circ \varphi_{-a}$.

Note $\varphi_{-a}(z) = \dfrac{z+a}{1+\bar{a}z} \Rightarrow \varphi_{-a}(0) = a$

$\Rightarrow g(0) = f(a) = 0$

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Moreover, g is an automorphism of $\mathbb{D}$

$\Rightarrow \exists\, h$: analytic $\mathbb{D} \to \mathbb{D}$ s.t $h \circ g(z) = z$

Differentiate $\Rightarrow h'(g(z)) \cdot g'(z) = 1$

Let $z=0 \Rightarrow |h'(0)||g'(0)| = 1$

But by Schwarz's Lemma, $|h'(0)| \geq 1$, $|g'(0)| \geq 1$

$$\Rightarrow |h'(0)| = 1, \quad |g'(0)| = 1$$

Thus by Schwarz's Lemma again, $g(z) = cz$ for some $|c|=1$

That is $f \circ \varphi_{-a}(z) = cz$

$$\Rightarrow f(z) = c\,\varphi_a(z)$$

Thm: Let f be an automorphism of $\mathbb{D}$. Assume $f(a) = \alpha$, where $a \in \mathbb{D}$, $\alpha \in \mathbb{D}$. Then

$$f(z) = \varphi_{-\alpha}(c\,\varphi_a(z))$$

Pf: Let $h \triangleq \varphi_\alpha \circ f$. Then h is an automorphism of $\mathbb{D}$ as well. and

$$h(a) = \varphi_\alpha(\alpha) = 0$$

By the above thm, $h = c\varphi_a$ for some $|c|=1$.

$$\Rightarrow \varphi_\alpha \circ f = c\,\varphi_a$$

$$\Rightarrow f = \varphi_{-\alpha}(c\,\varphi_a).$$

Thm: Let $g: \mathbb{D} \to \mathbb{D}$ analytic. Assume $g(a) = \alpha$, where $a \in \mathbb{D}$, $\alpha \in \mathbb{D}$. Then

(1) $$|g'(a)| \leq \frac{1-|\alpha|^2}{1-|a|^2}$$

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(2) "=" holds in (1) $\Leftrightarrow g = \varphi_{-\alpha}(c\varphi_a(z))$

Idea of Pf: Modify the map to make $0 \to 0$, then Schwarz's lemma.

Pf: Define $h \triangleq \varphi_\alpha \circ g \circ \varphi_{-a}$. Then

$$h(0) = \varphi_\alpha \circ g \circ \varphi_{-a}(0) = \varphi_\alpha \circ g(a) = \varphi_\alpha(\alpha) = 0.$$

Moreover, h maps $\mathbb{D}$ to $\mathbb{D}$.

By Schwarz's lemma, $|h'(0)| \le 1$

Note
$$\begin{aligned} h'(0) &= (\varphi_\alpha \circ g)'(\varphi_{-a}(0)) \cdot \varphi_{-a}'(0) \\ &= \varphi_\alpha'(g(a)) \cdot g'(a)(1-|a|^2) \\ &= \varphi_\alpha'(\alpha) \cdot g'(a)(1-|a|^2) \\ &= (1-|\alpha|^2)^{-1} \cdot g'(a)(1-|a|^2) \end{aligned}$$

$$\Rightarrow \quad |g'(a)| \frac{1-|a|^2}{1-|\alpha|^2} \le 1$$

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(2) Note $|g'(a)| = \frac{1-|\alpha|^2}{1-|a|^2} \Leftrightarrow |h'(0)| = 1.$

By Schwarz's lemma, we have this holds iff $h(z) = cz$ with $|c| = 1$

i.e. $\varphi_\alpha \circ g \circ \varphi_{-a}(z) = cz$

$$\Leftrightarrow \quad \varphi_\alpha \circ g(z) = c\varphi_a(z)$$

$$\Leftrightarrow \quad g(z) = \varphi_{-\alpha}(c\varphi_a(z))$$